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MECANICĂ TEORETICĂ
FIZICĂ**

1991

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Secția I

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Acad. Prof. Dr. **DIMITRIE I. MANGERON**
(1906—1991)



Professor Dr. Dimitrie I. Mangeron, member of the Romanian Academy, a leading figure of the Romanian science and high technical teaching, was born in 15/28 November 1906, in Kishinev. After brilliant studies, he graduated in Jassy the Faculty of Sciences of the University „Al. I. Cuza”. Then, after two years of more studies in Italy, he became doctor in mathematical sciences, with a remarkable work, devoted to some nonlinear differential equations with partial derivatives. These researches contain in germ what will be later called as „Mangeron equations”.

Returned at home, between 1935—1941, he covers all the university ranks, until that of professor. Since 1940 and for all his life, professor Mangeron works with an outstanding passion at the Polytechnic Institute „Gh. Asachi” in Jassy, rapidly becoming a great scientist and a great teacher. He delivered courses on mathematics, but

especially he remains the fascinating professor in theoretical mechanics, for countless series of students, from all the faculties. He enchanted his audience through a wide intellectual opening, striking the listener imagination by a great capacity to penetrate with him in the heart of the more actual scientific problems. For a long time he worked hardly as head of the Departments of Theoretical Mechanics and Mathematics.

During more than sixty years, until the last moment, professor Mangeron was an almost unique example of devotion in the service of teaching and science. With an contaminating enthusiasm and outstanding generosity, he created well-known schools of scientific researches. Open minded and exceptionally gifted, he contributed with remarkable and lasting results to the progress of mathematics and of theoretical and applied mechanics. Through most than 600 papers, written alone or with coworkers from 25 countries, professor Dimitrie Mangeron solved a great number of sharp scientific problems, very impressive due to their up-to-dateness, widtness and variety. He became very soon known and recognized by the international scientific community, who inceasingly asked him to present conferences, to deliver courses or to guide doctoral dissertations, all over the world.

Through his scientific researches, professor Mangeron put his mark on various fields of great interest, such as analytical mechanics, mechanics of vibration, theory of mechanisms and machines, optimal control in engineering, biomechanics, theory and applications of differential equations with partial derivatives, variational methods, history of mathematics and mechanics. Continuing his remarkable achievements, notorious specialists have introduced them in the international scientific circuit, as "Mangeron polyvibrating equations", "Mangeron operators", "Mangeron generalized equations in analytical mechanics", etc.

Scientist of enciclopedic style, professor Dimitrie Mangeron kept up a vast correspondence with an incredible number of scientists, editorial staffs of reviews, universities and research institutes from all the continents. Simultaneously, he spend incessantly much energy to spread over the world our scientific achievements and to keep up a lively exchange of scientific information. As nobody else, professor Mangeron contributed to make known the Polytechnic Institute „Gh. Asachi“. He was elected member of numerous scientific societies, academies, editorial staffs of prestigious international journals from Europe, Nord and South America and Asia.

One great achievement of professor Mangeron remains the foundation, in 1946, of the now well known „Bulletin of the Polytechnic Institute «Gh. Asachi», Jassy“. He succeeded in securing it a high scien-

tific level, who attracted very large contributions of numerous remarkable scientists from all geographical regions. Owing to professor Mangeron, our „Bulletin“ has acquired a high international prestige and a strong capacity to penetrate in the international scientific media, on the basis of material exchange with a wide range of similar issues.

Professor Dimitrie I. Mangeron, member of the Romanian Academy, is definitively entered into the first ranks of our great scientists and teachers of this century. He remains alive in the memory of all who know him and cannot forget his largely open mind to universality, his inexhaustible desire to know and skill to spread knowledge, his unquestionable competence and profound humanism.

Editorial Staff of the
„Bulletin of the Polytechnic
Institute «Gh. Asachi», Jassy“

SOME ORDERED GROUPS ON SOME CARTESIAN PRODUCTS

BY

GH. COSTOVICI

1. On some cartesian products of partially ordered (p. o.) groups, one obtain new p. o. groups by means of bilinear mappings and one give more examples.

2. If G_1, G_2, G are multiplicative groups, then the mapping $f: G_1 \times G_2 \rightarrow G$ is bilinear $\Leftrightarrow$ are fulfilled

$$(1) \quad \forall x, y \in G_1 \text{ and } \forall z \in G_2, \quad f(xy, z) = f(x, z)f(y, z);$$

$$(2) \quad \forall x \in G_1 \text{ and } \forall y, z \in G_2, \quad f(x, yz) = f(x, y)f(x, z).$$

If $f: G_1 \times G_2 \rightarrow G$ is bilinear then take place

$$(3) \quad \forall x \in G_1, \quad f(x, 1) = 1;$$

$$(4) \quad \forall y \in G_2, \quad f(1, y) = 1.$$

Indeed $f(x, 1) = f(x, 11) = f(x, 1)f(x, 1)$ and similarly.

Theorem (T). I. If $f_i: G_i \times G_i \rightarrow G, i = \overline{1, p}$ are bilinear mappings and G is commutative, then $H = G_1 \times \dots \times G_p \times G$ becomes group by the operation

$$(a_1, \dots, a_p, a_{p+1}) \circ (b_1, \dots, b_p, b_{p+1}) = (a_1 b_1, \dots, a_p b_p, a_{p+1} b_{p+1} f_1(a_1, b_1) \dots f_p(a_p, b_p)).$$

II. If besides the hypotheses of I, $G_i, i = \overline{1, p}$, and G are p. o. groups, then $(H, \circ)$ becomes p. o. group by the lexicographic order that is with the positive cone $P = \{(x_1, \dots, x_{p+1}) \in H; x_1 > 1 \text{ or } (x_1 = 1 \text{ and } x_2 > 1) \text{ or } \dots \text{ or } (x_1 = 1, \dots, x_{p-1} = 1 \text{ and } x_p > 1) \text{ or } (x_1 = 1, \dots, x_p = 1 \text{ and } x_{p+1} \geq 1)\}$.

Proof of I. $[(a_1, \dots, a_p, a_{p+1}) \circ (b_1, \dots, b_p, b_{p+1})] \circ (c_1, \dots, c_p, c_{p+1}) =$

$$= (a_1 b_1, \dots, a_p b_p, a_{p+1} b_{p+1} f_1(a_1, b_1) \dots f_p(a_p, b_p)) \circ (c_1, \dots, c_p, c_{p+1}) =$$

$$= (a_1 b_1 c_1, \dots, a_p b_p c_p, a_{p+1} b_{p+1} f_1(a_1, b_1) \dots f_p(a_p, b_p) c_{p+1} f_1(a_1 b_1, c_1) \dots$$

$$f_p(a_p b_p, c_p) = (a_1 b_1 c_1, \dots, a_p b_p c_p, a_{p+1} b_{p+1} f_1(a_1, b_1) \dots f_p(a_p, b_p) c_{p+1}$$

$$f_1(a_1, c_1) f_1(b_1, c_1) \dots f_p(a_p, c_p) f_p(b_p, c_p)).$$

$$(a_1, \dots, a_p, a_{p+1}) \circ [(b_1, \dots, b_p, b_{p+1}) \circ (c_1, \dots, c_p, c_{p+1})] = (a_1, \dots, a_p, a_{p+1}) \circ$$

$$\circ (b_1 c_1, \dots, b_p c_p, b_{p+1} c_{p+1} f_1(b_1, c_1) \dots f_p(b_p, c_p)) = (a_1 b_1 c_1, \dots, a_p b_p c_p,$$

$$\begin{aligned} a_{p+1}b_{p+1}c_{p+1}f_1(b_1, c_1) \dots f_p(b_p, c_p)f_1(a_1, b_1c_1) \dots f_p(a_p, b_pc_p) = \\ = (a_1b_1c_1, \dots, a_pb_pc_p, a_{p+1}b_{p+1}c_{p+1}f_1(b_1, c_1) \dots f_p(b_p, c_p) \\ f_1(a_1, b_1)f_1(a_1, c_1) \dots f_p(a_p, b_p)f_p(a_p, c_p)). \end{aligned}$$

G being commutative, the operation „ $\circ$ ” is associative.

$$\exists (1, \dots, 1) \in H \text{ so that } \forall (a_1, \dots, a_{p+1}) \in H \text{ we have } (a_1, \dots, a_{p+1}) (1, \dots, 1) = (a_1, \dots, a_{p+1}).$$

If $(a_1, \dots, a_p, a_{p+1}) \in H$, $\exists (a_1^{-1}, \dots, a_p^{-1}, a_{p+1}^{-1} f_1(a_1, a_1) \dots f_p(a_p, a_p)) \in H$ so that

$$\begin{aligned} (a_1, \dots, a_p, a_{p+1}) \circ (a_1^{-1}, \dots, a_p^{-1}, a_{p+1}^{-1} f_1(a_1, a_1) \dots f_p(a_p, a_p)) = \\ = (1, \dots, 1, 1 f_1(a_1, a_1) \dots f_p(a_p, a_p) f_1(a_1, a_1^{-1}) \dots f_p(a_p, a_p^{-1})) = \\ = (1, \dots, 1, f_1(a_1, 1) \dots f_p(a_p, 1)) = (1, \dots, 1). \end{aligned}$$

Proof of II. $P = \{(x_1, \dots, x_{p+1}) \in H; x_1 > 1 \text{ or } (x_1 = 1 \text{ and } x_2 > 1) \text{ or } \dots \text{ or } (x_1 = 1, \dots, x_{p-1} = 1 \text{ and } x_p > 1) \text{ or } (x_1 = 1, \dots, x_p = 1 \text{ and } x_{p+1} \geq 1)\}$.

We show that $P \cap P^{-1} = \{(1, \dots, 1)\}$. Let $(x_1, \dots, x_{p+1}) \in P \cap P^{-1}$ be. Then $(x_1, \dots, x_{p+1}) \in P$ and $(x_1, \dots, x_{p+1}) = (p_1, \dots, p_{p+1})^{-1} = (p_1^{-1}, \dots, p_p^{-1}, p_{p+1}^{-1} f_1(p_1, p_1) \dots f_p(p_p, p_p))$ with $(p_1, \dots, p_{p+1}) \in P$.

$x_1 > 1$ and $x_1 = p_1^{-1} \leq 1$, impossibly.

$x_1 = 1, x_2 > 1$ and $x_2 = p_2^{-1} \leq 1$, impossibly.

$x_1 = 1, \dots, x_{p-1} = 1, x_p > 1$ and $x_p = p_p^{-1} \leq 1$, impossibly.

$x_1 = 1 = p_1^{-1}, \dots, x_p = 1 = p_p^{-1}, x_{p+1} \geq 1$ and $x_{p+1} = p_{p+1}^{-1} f_1(1, 1) \dots f_p(1, 1) = p_{p+1}^{-1} \leq 1 \Rightarrow x_{p+1} = 1$.

We show that $P \circ P \subseteq P$. Let $(x_1, \dots, x_{p+1}) \in P$. Then $x_1 > 1$ or $(x_1 = 1$ and $x_2 > 1)$ or ... or $(x_1 = 1, \dots, x_{p-1} = 1$ and $x_p > 1)$ or $(x_1 = 1, \dots, x_p = 1$ and $x_{p+1} \geq 1)$. Let $(y_1, \dots, y_{p+1}) \in P$ be. Then $y_1 > 1$ or $(y_1 = 1$ and $y_2 > 1)$ or ... or $(y_1 = 1, \dots, y_{p-1} = 1$ and $y_p > 1)$ or $(y_1 = 1, \dots, y_p = 1$ and $y_{p+1} \geq 1)$. Let $t = (x_1, \dots, x_{p+1}) \circ (y_1, \dots, y_{p+1}) = (x_1 y_1, \dots, x_p y_p, x_{p+1} y_{p+1} f_1(x_1, y_1) \dots f_p(x_p, y_p))$ be.

$x_1 > 1$ and $y_1 > 1 \Rightarrow x_1 y_1 > 1 \Rightarrow t \in P$.

$x_1 > 1, y_1 = 1$ and $y_2 > 1 \Rightarrow x_1 y_1 > 1 \Rightarrow t \in P$.

$x_1 = 1, x_2 > 1$ and $y_1 > 1 \Rightarrow x_1 y_1 > 1 \Rightarrow t \in P$.

$x_1 = 1, x_2 > 1, y_1 = 1$ and $y_2 > 1 \Rightarrow x_1 y_1 = 1$ and $x_2 y_2 > 1 \Rightarrow t \in P$.

.....

$x_1 = 1, \dots, x_p = 1, x_{p+1} \geq 1, y_1 = 1, \dots, y_p = 1$ and $y_{p+1} \geq 1 \Rightarrow x_1 y_1 = 1, \dots, x_p y_p = 1$ and $x_{p+1} y_{p+1} f_1(x_1, y_1) \dots f_p(x_p, y_p) = x_{p+1} y_{p+1} \geq 1 \Rightarrow t \in P$.

We show that $\forall (z_1, \dots, z_{p+1}) \in H$ we have $(z_1, \dots, z_{p+1}) \circ P \circ (z_1, \dots, z_{p+1})^{-1} \subseteq P$. Let $(p_1, \dots, p_{p+1}) \in P$ be $\Leftrightarrow p_1 > 1$ or $(p_1 = 1$ and $p_2 > 1)$ or ... or $(p_1 = 1, \dots, p_{p-1} = 1$ and $p_p > 1)$ or $(p_1 = 1, \dots, p_p = 1$ and $p_{p+1} \geq 1)$. Let $l = (z_1, \dots, z_{p+1}) \circ (p_1, \dots, p_{p+1}) \circ (z_1^{-1}, \dots, z_{p+1}^{-1}) f_1(z_1, z_1) \dots f_p(z_p, z_p) = (z_1 p_1, \dots, z_p p_p, z_{p+1} p_{p+1}) f_1(z_1, p_1) \dots f_p(z_p, p_p) \circ (z_1^{-1}, \dots, z_p^{-1}, z_{p+1}^{-1}) f_1(z_1, z_1) \dots f_p(z_p, z_p) = (z_1 p_1 z_1^{-1}, \dots, z_p p_p z_p^{-1}, z_{p+1} p_{p+1} z_{p+1}^{-1}) f_1(z_1, p_1) \dots f_p(z_p, p_p) z_{p+1}^{-1} f_1(z_1, z_1) \dots f_p(z_p, z_p) f_1(z_1 p_1, z_1^{-1}) \dots f_p(z_p p_p, z_p^{-1}) = (z_1 f_1 z_1^{-1}, \dots, z_p p_p z_p^{-1}, p_{p+1} f_1(z_1, p_1) \dots f_p(z_p, p_p) f_1(p_1, z_1^{-1}) \dots f_p(p_p, z_p^{-1})).$

$$p_1 > 1 \Rightarrow z_1 p_1 z_1^{-1} > 1 \Rightarrow l \in P.$$

$$p_1 = 1 \text{ and } p_2 > 1 \Rightarrow z_1 p_1 z_1^{-1} = 1 \text{ and } z_2 p_2 z_2^{-1} > 1 \Rightarrow l \in P.$$

.....

$$p_1 = 1, \dots, p_{p-1} = 1 \text{ and } p_p > 1 \Rightarrow z_1 p_1 z_1^{-1} = 1, \dots, z_{p-1} p_{p-1} z_{p-1}^{-1} = 1 \text{ and } z_p p_p z_p^{-1} > 1 \Rightarrow l \in P.$$

$$p_1 = 1, \dots, p_p = 1 \text{ and } p_{p+1} \geq 1 \Rightarrow z_1 p_1 z_1^{-1} = 1, \dots, z_p p_p z_p^{-1} = 1 \text{ and } p_{p+1} f_1(z_1, p_1) \dots f_p(z_p, p_p) f_1(p_1, z_1^{-1}) \dots f_p(p_p, z_p^{-1}) = p_{p+1} \geq 1 \Rightarrow l \in P.$$

Examples. Let $\mathbb{C}$ be the additive group of complex numbers (or complex rational numbers). One verify that $f_i : \mathbb{C} \times \mathbb{C} \rightarrow \mathbb{R}$, $i = \overline{1, 8}$ where

$$f_1((a, b)(x, y)) = \operatorname{Re}((a, b)(x, y)) = ax - by,$$

$$f_2((a, b)(x, y)) = \operatorname{Re}((a, b)(x, -y)) = ax + by,$$

$$f_3((a, b), (x, y)) = \operatorname{Re}((a, -b)(x, y)) = ax + by = f_2((a, b), (x, y)),$$

$$f_4((a, b)(x, y)) = \operatorname{Re}((a, -b)(x, -y)) = ax - by = f_1((a, b)(x, y)),$$

$$f_5((a, b), (x, y)) = \operatorname{Im}((a, b)(x, y)) = ay + bx,$$

$$f_6((a, b), (x, y)) = \operatorname{Im}((a, b)(x, -y)) = -ay + bx,$$

$$f_7((a, b), (x, y)) = \operatorname{Im}((a, -b)(x, y)) = ay - bx,$$

$$f_8((a, b), (x, y)) = \operatorname{Im}((a, -b)(x, -y)) = -ay - bx, \text{ are bilinear.}$$

Then, by T, I, and by means of these 6 bilinear mappings, $H = \mathbb{C}^n \times \mathbb{R}$ becomes group in 6^n ways. If one consider $H = \mathbb{C}^n \times \mathbb{R}$ the group obtained this way using f_i of n times, we obtain Heisenberg's group from [2].

In [1, p. 16] one consider $\mathbb{C}$ partially ordered in 5 ways. Then, by T, I, II, $H = \mathbb{C}^n \times \mathbb{R}$ becomes p. o. group in $5^n 6^n$ ways.

Similarly can be considered complex rational numbers and $\mathbb{Q}$.

REFERENCES

1. Fuchs L., *Partially Ordered Algebraic Systems*. Pergamon Press, Oxford, 1963.
2. Müller D., *A Restriction Theorem for the Heisenberg Group*. Ann. of Math., **131**, 3 (May 1990).

UNELE GRUPURI ORDONATE PE UNELE PRODUSE CARTEZIENE

(Rezumat)

Pe unele produse carteziane de grupuri parțial ordonate se obțin, cu ajutorul aplicațiilor biliniare, structuri de grup parțial ordonat și se dau exemple.

ON THE D-AXIOM ON A STANDARD AUTODUAL H -CONE

BY

LILIANA POPA

We prove that on a standard autodual H -cone of functions the existence of a measure of representation, implies the axiom D. We use the notations of [1].

Let $S = S^{**}$ be a standard H -cone of functions on a saturated set X , which satisfies the axiom A. S is autodual relative to the H -isomorphism φ and 1 is substractible. Because S is standard it results that P , the cone of all I -potentials is standard and satisfies the axiom of convergence and nearly continuity. In this case the set of all the pure potentials S_P coincides with the set of all nearly continuous elements and

$$S_\Delta \subset P \subset S_P,$$

where $S_\Delta = \{s \in S \mid \varphi(s)(s) < \infty\}$.

For any $U \subset X$, naturally continuous, we denote by $\varphi_P(U)$ the set of all the pure potentials of $\varphi(U)$ and by $\varphi_H(U)$ the set of all the substractible elements of $\varphi(U)$. For any $f \in \varphi(U)$, of the form $f = h + p$, with $h \in \varphi_H(U)$ and $p \in \varphi_P(U)$, we define

$$\sigma_U(f) = \sigma_U(p),$$

where $\sigma_U(p)$ is the H -measure associated with $\bar{\varphi}(p)$ ($\bar{\varphi}$ is the H -isomorphism which gives the autoduality of $\varphi(U)$, [5]).

In [4] we have proved that these applications have the following properties :

1. $\sigma_U(f_1 + f_2) = \sigma_U(f_1) + \sigma_U(f_2)$, for any $f_1, f_2 \in \varphi(U)$,
 $\sigma_U(\alpha f) = \alpha \sigma_U(f)$, for any $\alpha \in \mathbb{R}_+$, $f \in \varphi(U)$.
2. For any $f, g \in \varphi(U)$, with $f \leq g$, the following equivalence is valid
 $f < g$ iff $\sigma_U(f) \leq \sigma_U(g)$.
3. For any $U_1 \subseteq U_2$, naturally open and for any $p \in \varphi_P(U_2)$,

$$\sigma_{U_1}(p)|_{U_1} = \sigma_{U_1}(p - \hat{B}p), \text{ where } \hat{B} = B^{U_1 \setminus U_1} \text{ (in } \varphi(U_2)).$$

We denote by $<, >$ the inner product from the Dirichlet space H and by $[\cdot, \cdot]$ the induced bilinear form.

Proposition 1. *If for any $f \in S_{\Delta} - S_{\Delta}$, $[f^+, f^-] = 0$, then for any $x, y \in H$, with $x \wedge y = 0$ we have*

$$\langle x, y \rangle = 0.$$

Proof. Let $x, y \in H$, with $x \wedge y = 0$. Then $x, y \in H_+$. There exist $x_n, y_n \in (S_{\Delta} - S_{\Delta})_+$, which converge in H to x and y respectively. Let

$$x'_n = X_n - x_n \wedge y_n = (x_n - y_n) \vee 0 = (x_n - y_n)^+ \in S_{\Delta} - S_{\Delta};$$

$$y'_n = y_n - x_n \wedge y_n = (y_n - x_n) \vee 0 = (x_n - y_n)^- \in S_{\Delta} - S_{\Delta}.$$

Because the lattice operations are continuous, it results that

$$(x_n - y_n)^+ \rightarrow (x - y)^+ = x - x \wedge y = x;$$

$$(x_n - y_n)^- \rightarrow (x - y)^- = y - x \wedge y = y.$$

We have $x'_n \wedge y'_n = 0$, too. It results that

$$\langle x'_n, y'_n \rangle = \langle (x_n - y_n)^+, (x_n - y_n)^- \rangle = [(x_n - y_n)^+, (x_n - y_n)^-] = 0.$$

Passing to the limit, we have

$$\langle x, y \rangle = \langle (x - y)^+, (x - y)^- \rangle = \lim_{n \rightarrow \infty} \langle x'_n, y'_n \rangle = 0.$$

From [1], we know that the previous condition implies the axiom D.

We give two situations which assure that $[f^+, f^-] = 0$, for any $f \in S_{\Delta} - S_{\Delta}$.

I. We suppose that the family of applications $\{\sigma_U\}_{U \subset X}$ is a measure of representation. That means that the following condition is valid

4. For any U_1, U_2 , naturally open, with $U_1 \subseteq U_2$ and any $f \in \varphi(U_2)$,

$$\sigma_{U_1}(f)|_{U_1} = \sigma_{U_1}(f|_{U_1}).$$

In [4], we have proved that the condition 4 is implied by the axiom D. If the family $\{\sigma_U\}$ is a measure of representation, we may construct a mutual gradient measure, with the supplementary property $\delta_{[f^+, f^-]} = 0$, for any f , naturally continuous, which locally is a difference of bounded elements of $\varphi(U)$. Indeed, if $f \in S_{\Delta} - S_{\Delta}$ is like before, then

$$\delta_{[f^+, f^-]}(1) = \frac{1}{2} \left(\int_X f^+ d\sigma(f^-) + \int_X f^- d\sigma(f^+) - \sigma(f^+ f^-)(1) \right) = [f^+, f^-].$$

Proposition 2. *If the family of applications satisfies the axioms 1, 2, 3, then the property 4 is equivalent with the following assertion: for any U_1, U_2 , naturally open, with $U_1 \subseteq U_2$ and $f \in \varphi(U_2)$ it results that $(B^{U_2 \setminus U_1} f)|_{U_1}$ is substractible in $\varphi(U_1)$.*

Proof. We suppose that $(B^{U_2 \setminus U_1} f)|_{U_1}$ is substractible. Let $h \in \varphi_H(U_2)$ and $p \in \varphi_P(U_2)$, such that $f = h + p$. We have

$$f|_{U_1} = h - B^{U_2 \setminus U_1} h + p - B^{U_2 \setminus U_1} p + B^{U_2 \setminus U_1} f|_{U_1}.$$

But $h - B^{U_2 \setminus U_1} h$ is substractible [4] and from hypothesis, $B^{U_2 \setminus U_1} f$ is substractible in $\varphi(U_1)$. We deduce that

$$\sigma_{U_1}(f|_{U_1}) = \sigma_{U_1}(p - B^{U_2 \setminus U_1} p) = \sigma_{U_1}(p)|_{U_1} = \sigma_{U_1}(f)|_{U_1}.$$

Conversely, we prove firstly that if p is from $\varphi_P(U_2)$, then $B^{U_2 \setminus U_1} p$ is subtractible in U_1 . From the hypothesis we get

$$\sigma_{U_1}(p|_{U_1}) = \sigma_{U_1}(p)|_{U_1} = \sigma_{U_1}(p - B^{U_2 \setminus U_1} p).$$

But $\sigma_{U_1}(p|_{U_1}) = \sigma_{U_1}(p - B^{U_2 \setminus U_1} p) + \sigma_{U_1}(B^{U_2 \setminus U_1} p|_{U_1})$. From the last two relations we deduce

$$\sigma_{U_1}(B^{U_2 \setminus U_1} p|_{U_1}) = 0.$$

Hence $B^{U_2 \setminus U_1} p$ is subtractible in $\varphi(U_1)$. If $h \in \varphi_H(U_2)$, we have $\sigma_{U_1}(h) = 0$. But $h|_{U_1} = h - B^{U_2 \setminus U_1} h + (B^{U_2 \setminus U_1} h)|_{U_1}$ and $h - B^{U_2 \setminus U_1} h$ is subtractible in $\varphi(U_1)$. We have

$$0 = \sigma_{U_1}(h)|_{U_1} = \sigma_{U_1}(h|_{U_1}) = \sigma_{U_1}((B^{U_2 \setminus U_1} h)|_{U_1}).$$

Hence $B^{U_2 \setminus U_1} h|_{U_1}$ is subtractible in $\varphi(U_1)$.

II. We give another situation, which assures that for any $f \in S_\Delta - S_\Delta$, $[f^+, f^-] = 0$. We recall from [4], that for any $p \in S_P$, the H -measure $\sigma(p)$ is carried by $\text{carr}_n p$ and suppose that the next hypothesis of approximation is fulfilled:

For any $f \in S_\Delta - S_\Delta$, naturally continuous and $\varepsilon > 0$, there exist p and q from S_P , such that

$$\begin{aligned} \text{carr}_n p &\subset [f < 0], \quad \text{carr}_n q \subset [f < 0]; \\ 0 &\leq p - q \leq f^- \leq p - q + \varepsilon. \end{aligned}$$

Proposition 3. For any f from $S_\Delta - S_\Delta$, with $\sigma(|f|)(1) < \infty$, naturally continuous, we have

$$[f^+, f^-] = 0.$$

Proof. $[f^+, f^-] = \int_X f^+ d\sigma(f^-) = \int_X f^- d\sigma(f^+)$. Let $\varepsilon > 0$ and p, q like in the previous hypothesis. Then

$$\begin{aligned} \left| \int_X (f^- - (p - q)) d\sigma(f^+) \right| &\leq \varepsilon \sigma(f^+)(1); \\ \int_X (p - q) d\sigma(f^+) &= \int_X f^+ d\sigma(p - q) = \int_{[f < 0]} f^+ d\sigma(p - q) = 0. \end{aligned}$$

Hence

$$\left| \int_X f^- d\sigma(f^+) \right| \leq \varepsilon \sigma(f^+)(1).$$

But ε is arbitrary and it results that $\int_X f^- d\sigma(f^+) = 0$.

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REFERENCES

1. Boboc N., Bucur Gh., Cornea A., *Order and Convexity in Potential Theory: H-Cones*. Lecture Notes in Math., 853, Springer, Berlin—Heidelberg—New York, 1981.
2. Boboc N., Bucur Gh., *Potentials and Pure Potentials in H-Cones*. Rev. Roum. Pures Appl., 27 (5), 529—549 (1982).

3. Maeda F. Y., *Dirichlet Integrals on Harmonic Spaces*. Lecture Notes in Math., **803**, Springer, Berlin—Heidelberg—New York, 1980.
4. Popa L., *Dirichlet Integrals on Standard H -Cones*. Bull. Math. de la Soc. Sci. Math. Roum., **31 (79)**, 2, 153—161 (1987).
5. Popa L., *The Localization of the Autoquality Condition*. Bul. Inst. Polit. Iași, supl. Cercetări Matematice, 73—75. (1985).

ASUPRA AXIOMEI D-PE UN H -CON AUTODUAL STANDARD

(Rezumat)

Se arată că existența unei măsuri de reprezentare pe un H -con standard autodual antrenează axioma D. Implicația inversă a fost stabilită în [4].

REMARQUES SUR LES ESPACES T -MÉTRIQUES COMPLETS

PAR

TEMISTOCLE BÎRSAN

1. Introduction. Les espaces T -métriques généralisent les espaces métriques (classiques) dans le sens suivant :

Définition 1. On dit que $d: X \times X \rightarrow \mathbb{R}_+$ est une métrique sur X par rapport à la fonction $T: \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ si

I. $d(x, y) = 0 \Leftrightarrow x = y$,

II. $d(x, y) = d(y, x)$ pour tous $x, y \in X$,

III. $d(x, z) \leq T(d(x, y), d(y, z))$ pour tous $x, y, z \in X$, $x \neq y \neq z \neq x$. Dans ce cas, les fonctions T et d sont appelées *t-fonction* (c-à-d. fonction triangulaire) et *T -métrique*, respectivement. Le triplet (X, d, T) est appelé *espace T -métrique*.

L'étude des espaces T -métriques a été commencée dans [1] et continuée dans [2], [3], etc. Rappelons un minimum de résultats dont nous aurons besoin.

Théorème A [2]. Soit (X, d, T) un espace T -métrique dont la *t-fonction* T satisfait à la condition

$$\lim_{a \rightarrow 0, b \rightarrow 0} T(a, b) = 0.$$

Alors l'espace donné est métrisable.

Théorème B [2]. Soit (X, d, T) un espace T -métrique de sorte que

1° $T(0, b) \leq b$, pour tout $b \in \mathbb{R}_+$.

2° $t \mapsto T(t, b)$ est une application scs (c-à-d. semi-continue supérieurement) au point $t=0$ uniformément par rapport à $b \in K$, quel que soit l'ensemble compact $K \subset \mathbb{R}_+$.

Alors l'espace (X, d, T) est métrisable.

Dans ce travail on donne une généralisation du théorème de Cantor qui caractérise la complétude des espaces métriques et puis on l'applique dans la théorie des points fixes.

2. Espaces T -métriques complets. Soit (X, d, T) un espace T -métrique et A une partie nonvide de X . Désignons par $d(A)$ le diamètre de A , c-à-d. $d(A) = \sup\{d(x, y) ; x, y \in A\}$. Nous allons démontrer un résultat élémentaire mais important dans ce qui suit.

Proposition 1. Soit (X, d, T) un espace T -métrique dont la *t-fonction* T satisfait aux conditions 1° et 2°. Alors

(1)

$$d(\tilde{A}) = d(A).$$

Démonstration. Grâce au Théorème B, l'espace (X, d, T) est métrisable. Évidemment,

$$(2) \quad d(A) \leq d(\bar{A}).$$

Si $d(A) = +\infty$, alors de (2) résulte que $d(\bar{A}) = +\infty$ et, donc, (1) est vraie.

Supposons que $d(A) < +\infty$, c-à-d. A est une partie bornée. Soient $\varepsilon \in (0, 1)$ et $K = [0, d(A) + 1]$. Dans les hypothèses 1° et 2° il existe $\delta > 0$ tel que

$$(3) \quad T(t, b) < b + \varepsilon, \text{ quels que soient } t < \delta \text{ et } b \in K.$$

Si $x, y \in \bar{A}$, alors il y a $x_0, y_0 \in A$ de sorte que $d(x, x_0) < \delta$ et $d(y, y_0) < \delta$. Alors $d(x, y_0) \leq T(d(x, x_0), d(x_0, y_0))$ et, en utilisant (3), on peut conclure que $d(x, y_0) < d(x_0, y_0) + \varepsilon$ ou bien,

$$(4) \quad d(x, y_0) < d(A) + \varepsilon.$$

D'autre part, $d(x, y) = d(y, x) \leq T(d(y, y_0), d(y_0, x))$ et, en utilisant encore une fois la relation (3), on a $d(x, y) < d(x, y_0) + \varepsilon$, ou, en vertu de (4),

$$(5) \quad d(x, y) < d(A) + 2\varepsilon.$$

De (5), on suit que $d(\bar{A}) < d(A) + 2\varepsilon$ et, donc

$$(6) \quad d(\bar{A}) \leq d(A).$$

Les relations (2) et (6) nous conduisent à (1), c.q.f.d.

Le résultat suivant généralise un théorème connu de Cantor. La démonstration suit la voie classique. Nous mettons en évidence où interviennent les hypothèses sur la t -fonction T .

Théorème 1. Soit (X, d, T) un espace T -métrique vérifiant les conditions 1° et 2°. Alors (X, d, T) est complet si et seulement si pour toute suite $(F_n)_n$ telle que : a) F_n est une partie nonvide et fermée, $n \in \mathbb{N}$; b) $F_n \supset F_{n+1}$, $n \in \mathbb{N}$, et c) $\lim_n d(F_n) = 0$ il y a $x_0 \in \bigcap_n F_n$ et le point x_0 est unique.

Démonstration. Passons directement à la deuxième partie du théorème. Soit $(x_n)_n$ une suite de Cauchy de l'espace (X, d, T) . Pour établir que $(x_n)_n$ converge, considérons $A_n = \{x_n, x_{n+1}, \dots\}$ pour tout $n \in \mathbb{N}$ et la suite $(F_n)_n$ définie par $F_n = \bar{A}_n$, $n \in \mathbb{N}$. Il est évident que $(F_n)_n$ vérifie a) et c). Grâce aux hypothèses 1° et 2° relativement à T on a $d(F_n) = d(A_n)$. Mais $d(A_n) \rightarrow 0$, car $(x_n)_n$ est une suite de Cauchy. Par conséquent, $d(F_n) \rightarrow 0$ et la condition c) est vérifiée. Alors il existe $x_0 \in \bigcap_n F_n$. Comme $d(x_n, x_0) \leq d(F_n)$, il résulte que $x_n \rightarrow x_0$. Pour prouver l'unicité de x_0 , on suppose qu'il existe encore un point $x_1 \in \bigcap_n F_n$. Alors $d(x_0, x_1) \leq d(F_n)$, $n \in \mathbb{N}$ et donc $d(x_0, x_1) = 0$, c-à-d. $x_0 = x_1$.

3. Théorèmes de point fixe. Soient (X, d, T) un espace T -métrique et $f: X \rightarrow X$ une application quelconque. En suivant [4], on fait les notations : $I_b(f) = \{A \subset X; A \neq \emptyset, f(A) \subset A, d(A) < +\infty\}$, $I_{bf}(f) = \{A \in I_b(f); A = \bar{A}\}$, $O(x, f) = \{x, f(x), f^2(x), \dots, f^n(x), \dots\}$.

On dit que l'application f est une φ -contraction généralisée de l'espace (X, d, T) si

$$(7) \quad d(f(A)) \leq \varphi(d(A)), \quad A \in I_b(f),$$

où $\varphi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ satisfait aux conditions : (i) φ est scs ; (ii) φ est croissante ; (iii) $\varphi(r) < r$ pour tout $r > 0$.

Le résultat suivant est connu et se démontre facilement.

Lemme 1. Si φ satisfait aux conditions (i) – (iii) et $r > 0$, alors $\lim_{n \rightarrow \infty} \varphi^n(r) = 0$,

où $\varphi^n = \varphi \circ \varphi \circ \dots \circ \varphi$ (n fois).

Théorème 2. Soit (X, d, T) un espace T -métrique complet dont la t -fonction T vérifie les conditions 1° et 2°. Si $f: X \rightarrow X$ est une φ -contraction généralisée et $I_{bf}(f) \neq \emptyset$, alors l'application f a un point fixe unique.

Démonstration. En partant d'un ensemble $A \in I_{bf}(f)$, construisons la suite $(A_n)_{n \geq 1}$ par récurrence : $A_1 = \overline{f(A)}$, $A_{n+1} = \overline{f(A_n)}$. On établit que $A_{n+1} \subset A_n \subset A$, $A_n \in I_{bf}(f)$ et, en vertu de (7), $d(A_n) \leq \varphi^n(d(A))$, $n \geq 1$. Alors, grâce au Lemme 1, $\lim_{n \rightarrow \infty} d(A_n) = 0$. La t -fonction T vérifiant les conditions

1° et 2° et l'espace (X, d, T) étant complet, on peut appliquer le Théorème 1 relativement à la suite $(A_n)_{n \geq 1}$ et on conclut qu'il existe $x_0 \in X$ tel que $\{x_0\} = \bigcap_{n \geq 1} A_n$. Par conséquent, $f(x_0) = x_0$. Supposons maintenant que f admet deux points fixes x_0 et x_1 . On désigne $A = \{x_0, x_1\}$ et on constate que $f(A) = A$ et $d(A) \leq \varphi(d(A))$, donc $d(A) = 0$, c-à-d. $x_0 = x_1$, c.q.f.d.

On dit qu'un élément $x \in X$ est régulier pour f , si $d(O(x, f)) < +\infty$.

Lemme 2. Soit (X, d, T) un espace T -métrique dont la t -fonction T vérifie la condition de monotonie

$$3^\circ \quad T(a, b) \leq T(a, b') \text{ dès que } b \leq b'.$$

Si x est un élément régulier de f , alors, quel que soit $y \in X$, on a $d(A) < +\infty$, où $A = O(x, f) \cup \{y\}$.

Démonstration. En effet, $d(A) = \max\{d(O(x, f)), \sup_{a \in O(x, f)} d(y, a)\}$.

Mais

$$d(y, a) \leq T(d(y, x), d(x, a)) \leq T(d(x, y), d(O(x, f)))$$

et, donc, $\sup\{d(y, a); a \in O(x, f)\} < +\infty$. D'ici et du fait que $d(O(x, f)) < +\infty$, on a que $d(A) < +\infty$.

Théorème 3. Soit (X, d, T) un espace T -métrique complet dont la t -fonction T satisfait aux conditions 1°, 2° et 3°. Si $f: X \rightarrow X$ est une φ -contraction généralisée et $I_{bf}(f) \neq \emptyset$, alors on a :

1) f admet un point fixe unique x_0 ;

2) $x \in X$ étant un élément régulier de f , la suite des approximations successives $(f^n x)_n$ converge vers x_0 .

Démonstration. En vertu du Théorème 2, il nous reste à prouver l'affirmation 2). Dans ce but, soit $A_n = O(f^n x, f) \cup \{x_0\}$, pour tout $n \in \mathbb{N}$. Vu le Lemme 2, on a $d(A_n) < +\infty$. De même, on voit sans peine que $A_{n+1} \subset A_n$ et $f(A_n) = A_{n+1}$, $n \in \mathbb{N}$, donc $A_n \in I_b(f)$, $n \in \mathbb{N}$. Par conséquent, $d(A_n) \leq \varphi^n(d(A_0))$, $n \in \mathbb{N}$. Il suit que $\lim_{n \rightarrow \infty} d(A_n) = 0$ et, donc, $\lim_{n \rightarrow \infty} f^n x = x_0$, ce qu'il fallait démontrer.

Remarque. Comme dans le cadre des espaces métriques classiques [4, Corollaire 1], dans l'énoncé du Théorème 3 on peut remplacer la condition $I_b(f) \neq \emptyset$ par celle moins restrictive $I_b(f) \neq \emptyset$, si la fonction f est continue.

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BIBLIOGRAPHIE

1. Bîrsan T., *Une application d'un théorème de J. Aczél*. Publ. Math. (Debrecen), 31, 113–118 (1984).
2. Bîrsan T., *Quelques propriétés des espaces T-métriques*. Bul. Inst. Polit. Iași, XXXII (XXXVI), 1–4, s. I, 19–23 (1986).
3. Bîrsan T., *Théorèmes de type Niemytzki-Edelstein dans les espaces T-métriques*. Bul. Inst. Polit. Iași, XXXV (XXXIX), 3–4, s. I, 15–19 (1989).
4. Rus I.A., *Generalized φ -contractions*. Mathematica, 24(47), 1–2, 175–178 (1982).

OBSERVAȚII ASUPRA SPAȚIILOR T-METRICE COMPLETE

(Rezumat)

Se dă o generalizare a teoremei lui Cantor ce caracterizează completitudinea spațiilor T-metrice și apoi se utilizează rezultatul obținut pentru a stabili teoreme de punct fix relativ la φ -contracții generalizate.

DEGENERATED HYPERSURFACES OF SEMI-RIEMANNIAN MANIFOLDS

BY

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0. Introduction. The theory of hypersurfaces of Riemannian (semi-Riemannian) manifolds is as old as the differential geometry is, since the latter was first concerned with theory of surfaces in 3-dimensional Euclidean spaces. That is why we have a wide literature and important results on the subject (see B. Y. Chen [3], B. O'Neill [7] and M. Dajczer [4] and the references therein). However, in case the ambient manifold is not Riemannian but semi-Riemannian and the hypersurface is degenerate we have a few results and they are mostly concerned with degenerate hypersurfaces of a Minkowski space (see W. B. Bonnor [1], Fr. Cagnac [2], J. B. Kammerer [5] and R. Roșca [8], [9]).

It is the purpose of the present paper to initiate a study of the differential geometry of degenerate hypersurfaces, in its whole generality.

First, we introduce into the study the so-called screen distribution and obtain the decompositions (1.1) and (1.6). Then we get Gauss-Codazzi equations of the immersion of a degenerate hypersurface and characterizations of integrable and parallel screen distributions. Finally, we study totally geodesic and totally umbilical degenerate hypersurfaces of real space forms.

1. The Induced Geometrical Objects on a Degenerate Hypersurface.

Let $(\bar{M}, \bar{g})$ be an $(m+1)$ -dimensional semi-Riemannian manifold of index $\nu \in \{1, \dots, m\}$ where $m \geq 2$. Suppose M is a degenerate hypersurface of $\bar{M}$, that is the restriction g of $\bar{g}$ to the tangent bundle TM of M is degenerate. Thus, both the tangent space $T_x M$ and the normal space $T_x M^\perp$ are degenerate for each $x \in M$. Moreover, it is easy to see that $T_x M^\perp \subset T_x M$, and hence the normal bundle $TM^\perp$ (from the classical theory of Riemannian submanifolds) becomes now a null distribution on M .

Throughout the paper we suppose all manifolds be paracompact and smooth. We denote by $F(M)$ the algebra of differentiable functions on M and by $F(E)$ the $F(M)$ -module of differentiable sections of a vector bundle E over M . Also, all structures on M are assumed smooth.

Next, we consider a supplementary distribution SM to $TM^\perp$ in TM . Taking into account that $TM^\perp$ is orthogonal to TM , hence in particular it is orthogonal to SM , we get the orthogonal decomposition

$$(1.1) \quad TM = SM \perp TM^\perp.$$

We note that there always exists such a distribution since M is paracompact. We call SM the *screen distribution* of M .

Now, we consider a local section ξ of $TM^\perp$ and denote by h the projection of TM on SM with respect to (1.1). Then we have

$$(1.2) \quad X = hX + \eta(X)\xi,$$

where η is a differential 1-form locally defined on M .

Proposition 1. *The screen distribution is non-degenerate with respect to g .*

Proof. Suppose $g(hX, hY) = 0$ for any $Y \in \Gamma(TM)$ and by using (1.2) we get $g(Y, X) = 0$. Hence $X \in \Gamma(TM^\perp)$, which implies $hX = 0$ and the assertion is proved.

Since SM is non-degenerate, there exists a unique complementary orthogonal vector bundle $SM^\perp$ over M with 2-dimensional fibres, and thus we have the orthogonal decomposition

$$(1.3) \quad T\bar{M} = SM \perp SM^\perp.$$

Taking into account that ξ is a lightlike section of $SM^\perp$ and each fiber $SM_x^\perp$ is a non-degenerate plane, it is easy to see that there exists a unique lightlike vector field $N \in \Gamma(SM^\perp)$ such that

$$(1.4) \quad g(\xi, N) = 1.$$

Moreover, by (1.4) we see that N is not tangent to M and belongs to a 1-dimensional vector subbundle NM of $SM^\perp$. Hence we have

$$(1.5) \quad SM^\perp = TM^\perp \oplus NM,$$

where $\oplus$ means direct sum but not orthogonal. Finally, by using (1.1), (1.3) and (1.5) we obtain

$$(1.6) \quad T\bar{M} = SM \perp (TM^\perp \oplus NM) = TM \oplus NM.$$

Examples. 1. We denote by R_v^{m+1} the semi-Euclidean space with the semi-Riemannian metric

$$g(x, y) = - \sum_{i=1}^v x^i y^i + \sum_{j=v+1}^{m+1} x^j y^j.$$

Denote by q the quadratic form associated to g and consider M as the lightlike cone of R_v^{m+1} , that is, $M = q^{-1}(0) - \{0\}$. Then M is a degenerate hypersurface with $\xi = (x^1, \dots, x^{m+1})$.

2. Consider $\bar{M}$ as the Minkowski space R_1^{m+1} and M as one of the hyperplanes given by the equation

$$\sum_{i=2}^{m+1} \alpha_i x^i - \beta x^1 = 0; \quad \beta = \sqrt{\sum_{i=1}^{m+1} (\alpha_i)^2}; \quad (\alpha_2, \dots, \alpha_m) \neq (0, \dots, 0).$$

Then M is a degenerate hypersurface with $\xi = (\beta, \alpha_2, \dots, \alpha_m)$.

3. Let $\bar{M}$ be the space time R_1^4 and M be the hypersurface given by the parametric equations

$$\begin{aligned}x^1 &= u^1 chu^2 + shu^2, & x^2 &= u^1 + u^2, & x^3 &= u^1 shu^2 + chu^2, \\x^4 &= u^3, & u^1 &\neq 0.\end{aligned}$$

Then M is a degenerate hypersurface with $\xi = (chu^2, 1, shu^2, 0)$.

2. The Gauss-Codazzi Equations for Degenerate Hypersurfaces. Let $\bar{M}$ be a degenerate hypersurface of $\bar{M}$ and $\bar{\nabla}$ be the Levi-Civita connection on $\bar{M}$. Then by using the second form of the decomposition (1.6) we obtain

$$(2.1) \quad \bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N,$$

and

$$(2.2) \quad \bar{\nabla}_X N = -A_N X + \tau(X)N,$$

for any $X, Y \in \Gamma(TM)$; where $\nabla_X Y$ and $A_N X$ belong to $\Gamma(TM)$, while $B(X, Y)$ and $\tau(X)$ belong to $F(M)$. It is easy to check that ∇ is a torsion-free linear connection on M , and B, A_N and τ are a symmetric bilinear form on $\Gamma(TM)$, a linear operator on $\Gamma(TM)$ and a 1-form, respectively, locally defined on M . In fact, if we consider another system of coordinates where we choose $\{\xi^*, N^*\}$ as above, then we have

$$(2.3) \quad \xi^* = \alpha \xi; \quad N^* = \frac{1}{\alpha} N,$$

which, according to (2.1) and (2.2), implies

$$(2.4) \quad B^* = \alpha B, \quad A_N = \alpha A_{N^*}, \quad \alpha \tau(X) = \alpha^* \tau^*(X) + X(\alpha), \quad \forall X \in \Gamma(TM),$$

where α is a differentiable function nowhere zero on the intersection of the coordinates neighborhoods.

According to the general theory of submanifolds we call B and A_N the *second fundamental form* and the *shape operator* of the degenerate immersion of M in $\bar{M}$. Also, we call (2.1) and (2.2) the *Gauss* and *Weingarten formulae*, respectively. The following remarks are important for the whole study.

Remark 1. In general, the induced linear connection ∇ is not a metric connection. More precisely, we have

$$(2.5) \quad (\nabla_X g)(Y, Z) = B(X, Y)\eta(Z) + B(X, Z)\eta(Y),$$

for any $X, Y, Z \in \Gamma(TM)$, where η is the same 1-form form (1.2), and thus is given by

$$(2.6) \quad \eta(X) = g(X, N), \quad \forall X \in \Gamma(TM).$$

Remark 2. The study of the degenerate immersion of M in $\bar{M}$ depends on a particular choice of the screen distribution. However, the second fundamental form is invariant with respect to the change of the screen distribution, because from (2.1) we have

$$(2.7) \quad B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi), \quad \forall X, Y \in \Gamma(TM).$$

Next, for a complete study of the differential geometry of the degenerate hypersurface M of $\bar{M}$, taking consideration of the decomposition (1.1) we set

$$(2.8) \quad \nabla_X hY = \nabla_X^* hY + C(X, hY)\xi,$$

and

$$(2.9) \quad \nabla_X \xi = -A_X X + \epsilon(X)\xi,$$

for any $X, Y \in \Gamma(TM)$, where $\nabla_X^* hY$ and $A_X X$ belong to $\Gamma(SM)$ while $C(X, hY)$ and $\epsilon(X)$ belong to $F(M)$. Taking into account that $\bar{\nabla}$ is a metric connection and by using (1.4), (2.1), (2.2) and (2.9) we obtain

$$\epsilon(X) = \bar{g}(\nabla_X \xi, N) = \bar{g}(\bar{\nabla}_X \xi, N) = -\bar{g}(\xi, \bar{\nabla}_X N) = -\tau(X).$$

Hence (2.8) becomes

$$(2.10) \quad \nabla_X \xi = -A_X X - \tau(X)\xi.$$

It also follows that C is $F(M)$ -bilinear on $\Gamma(TM) \times \Gamma(SM)$ but, in general, it is not symmetric on $\Gamma(SM) \times \Gamma(SM)$, A_ξ is a tensor field of type (1.1) and ∇^* is a linear connection on the screen distribution.

Remark 3. Though the induced connection ∇ on M is not a metric connection, the linear connection ∇^* is a metric one on the screen distribution. In fact, by using (2.1) and (2.7) we obtain $(\nabla_X g)(hY, hZ) = (\bar{\nabla}_X g)(hY, hZ) = 0$, since $\bar{\nabla}$ is the Levi-Civita connection on $\bar{M}$. We note that C and A_ξ are locally defined and they change as follows

$$(2.11) \quad C = \alpha C^*; \quad A_{\xi^*} = \alpha A_\xi,$$

with respect to (2.3). We call C and A_ξ the second fundamental form and the shape operator, respectively, of SM . In case SM is integrable and we restrict C to $\Gamma(SM) \times \Gamma(SM)$ and A_ξ to $\Gamma(SM)$, they become the second fundamental form and the shape operator, respectively, of an arbitrary leaf of SM .

Now, by using (2.1), (2.2), (2.8) and (2.10) we obtain

$$(2.12) \quad g(A_N X, hY) = C(X, hY); \quad g(A_N X, N) = 0,$$

and

$$(2.13) \quad g(A_\xi X, hY) = B(X, hY); \quad g(A_\xi X, N) = 0,$$

for any $X, Y \in \Gamma(TM)$. Also, from $\bar{g}(\bar{\nabla}_N \xi, \xi) = 0$, by using (2.7) we get

$$(2.14) \quad B(X, \xi) = 0, \quad \forall X \in \Gamma(TM).$$

The second equation in (2.12) and (2.13) says that both A_N and A_ξ are $\Gamma(SM)$ -valued. Moreover, from (2.13) and (2.14) it follows that A_ξ is a symmetric operator on $\Gamma(SM)$ with respect to g and

$$(2.15) \quad A_\xi \xi = 0.$$

Further, by direct calculations, using (2.1) and (2.2) we obtain

$$(2.16) \quad \bar{R}(X, Y)Z = \{R(X, Y)Z + B(X, Z)A_N Y - B(Y, Z)A_N X\} + \\ + \{(\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \tau(X)(B(Y, Z) - \tau(Y)B(X, Z))\}N,$$

for any $X, Y, Z \in \Gamma(TM)$, where $\bar{R}$ and R are the curvature tensor fields of $\bar{M}$ and M , respectively, and we set

$$(2.17) \quad (\nabla_X B)(Y, Z) = X(B(Y, Z)) - B(\nabla_X Y, Z) - B(Y, \nabla_X Z).$$

From (2.16), taking account of (1.4), (2.12), and (2.13) it follows

$$(2.18) \quad \bar{g}(\bar{R}(X, Y)Z, hW) = g(R(X, Y)Z, hW) + B(X, Z)C(Y, hW) - \\ - B(Y, Z)C(X, hW),$$

$$(2.19) \quad \bar{g}(\bar{R}(X, Y)Z, \xi) = (\nabla_X B)(Y, Z) - (\nabla_Y B)(X, Z) + \\ + \tau(X)B(Y, Z) - \tau(Y)B(X, Z),$$

and

$$(2.20) \quad \bar{g}(\bar{R}(X, Y)Z, N) = g(R(X, Y)Z, N).$$

Next, by using (2.8) and (2.10) in the right hand side of (2.20) we obtain

$$(2.21) \quad \bar{g}(\bar{R}(X, Y)hZ, N) = (\nabla_X C)(Y, hZ) - (\nabla_Y C)(X, hZ) + \\ + \tau(Y)C(X, hZ) - \tau(X)C(Y, hZ),$$

and

$$(2.22) \quad \bar{g}(\bar{R}(X, Y)\xi, N) = C(Y, A_\xi X) - C(X, A_\xi Y) - 2d\tau(X, Y),$$

where we put

$$(2.23) \quad (\nabla_X C)(Y, hZ) = X(C(Y, hZ)) - C(\nabla_X Y, hZ) - C(Y, \nabla_X hZ),$$

and

$$(2.24) \quad d\tau(X, Y) = \frac{1}{2} \{X(\tau(Y)) - Y(\tau(X)) - \tau[X, Y]\}.$$

According to the general theory of submanifolds we call (2.18) the *equation of Gauss* and (2.19), (2.21) and (2.22) the *equations of Codazzi* for the degenerate immersion of M in $\bar{M}$.

3. The Screen Distribution on a Degenerate Hypersurface. Let M be a degenerate hypersurface of a semi-Riemannian manifold $\bar{M}$. Then by using (1.2), (2.6) and (2.8) and taking into account that ∇ is a torsion-free connection, we obtain

$$(3.1) \quad C(hX, hY) - C(hY, hX) = \eta([hX, hY]), \quad \forall X, Y \in \Gamma(TM).$$

Hence by (2.12) and (3.1) we have

Theorem 1. *Let M be a degenerate hypersurface of the semi-Riemannian manifold $\bar{M}$. Then the screen distribution SM is integrable if and only if one of the following conditions is fulfilled:*

(i) *The second fundamental form C of SM is symmetric on $\Gamma(SM)$.*

(ii) *The shape operator A_N of the immersion of M in $\bar{M}$ is symmetric with respect to the metric g .*

Now, we say that SM is a parallel distribution with respect to ∇ if we have

$$(3.2) \quad \nabla_X hY \in \Gamma(SM), \quad \forall X, Y \in \Gamma(TM).$$

Since ∇ is a torsion-free connection on M it follows that SM is integrable. Moreover, from (2.8) and (2.12) we obtain

Theorem 2. *The screen distribution is parallel if and only if one of the following conditions is satisfied:*

$$(3.3) \quad C(X, hY) = 0, \quad \forall X, Y \in \Gamma(TM),$$

or

$$(3.4) \quad A_N X = 0, \quad \forall X \in \Gamma(TM).$$

K. Nomizu and U. Pinkall [6] have given an example of affine immersion which they called the *graph immersion*. We recall it here. Suppose $\psi: (M, \nabla) \rightarrow R^m$ is an affine immersion such that ∇ is a flat connection. Consider R^m as a hyperplane of R^{m+1} and let N be a parallel vector field transversal to R^m . Then for any differentiable function $F: M \rightarrow R$ we define $f: M \rightarrow R^{m+1}$, $f(x) = \psi(x) + F(x)N$, for any $x \in M$. Thus, f is an affine immersion with $A_N = 0$ and it is called the graph immersion with respect to F .

Theorem 3 (K. Nomizu and U. Pinkall [6]). *Let $f: (M, \nabla) \rightarrow R^{m+1}$ be an affine immersion such that ∇ is a flat connection and $A_N = 0$. Then it is affinely equivalent to the graph immersion for a certain function $F: M \rightarrow R$.*

Now, since SM is non-degenerate we may consider an orthonormal basis $\{E_1, \dots, E_{m-1}\} \subset \Gamma(SM)$. Then by using (2.18), (2.20) and (3.3) we obtain

$$(3.5) \quad R(X, Y)Z = \sum_{i=1}^{m-1} \{\bar{g}(\bar{R}(X, Y)Z, E_i)E_i\} + \eta(\bar{R}(X, Y)Z)\xi,$$

for any $X, Y, Z \in \Gamma(TM)$. Hence in case SM is parallel, the curvature of M is completely determined by the curvature of $\bar{M}$. Moreover, by combining Theorems 2 and 3 and taking account of (3.5) we obtain

Theorem 4. *Let M be a degenerate hypersurface of R^{m+1} with parallel screen distribution. Then the immersion of M is affinely equivalent to the graph immersion of a certain function $F: M \rightarrow R$.*

Finally, we consider $\bar{M}$ as a semi-Riemannian manifold of constant curvature c . Then the curvature tensor of $\bar{M}(c)$ is given by

$$(3.6) \quad \bar{R}(U, V)W = c\{\bar{g}(U, W)V - \bar{g}(V, W)U\}, \quad \forall U, V, W \in \Gamma(TM).$$

Thus (3.5) and (3.6) imply

$$(3.7) \quad R(X, Y)Z = c\{g(X, Z)Y - g(Y, Z)X\}, \quad \forall X, Y, Z \in \Gamma(TM).$$

Hence we state

Theorem 5. *Let M be a degenerate hypersurface of $\bar{M}(c)$ with parallel distribution SM . Then all leaves of SM are semi-Riemannian manifolds of constant curvature c .*

4. Totally Geodesic and Totally Umbilical Degenerate Hypersurfaces.

Let M be a degenerate hypersurface of a semi-Riemannian manifold $\bar{M}$. Then

we say that M is *totally geodesic* in $\bar{M}$ if $B=0$ on M . According to (2.4) and (2.7) it follows that totally geodesic immersion neither depends on vector field ξ nor on the screen distribution SM . Then by using (2.1), (2.5) (2.10) and (2.13) we obtain

Theorem 6. *Let M be a degenerate hypersurface of $\bar{M}$. Then M is totally geodesic if and only if one of the following conditions is satisfied:*

- (i) *Each path of M with respect to ∇ is a geodesic of $\bar{M}$.*
- (ii) *The induced linear connection ∇ is a metric connection.*
- (iii) *The shape operator of the screen distribution vanishes identically on M .*
- (iv) *The vector field ξ is a Killing vector field on M .*
- (v) *The null distribution $TM^\perp$ is parallel with respect to ∇ .*

We now prove

Proposition 2. *Let M be a degenerate hypersurface of $\bar{M}$. Then there exists a local vector field $\xi' \in \Gamma(TM^\perp)$ whose integral curves are geodesics on both M and $\bar{M}$.*

Proof. First, by using (2.1), (2.10), (2.14) and (2.15) we obtain

$$\bar{\nabla}_\xi \xi = \nabla_\xi \xi = -\tau(\xi)\xi.$$

Then we consider a differentiable function f which is a solution of the equation with partial derivatives of first order

$$\xi(f) = \tau(\xi)f,$$

and take the vector field $\xi' = f\xi$. Then we obtain

$$\bar{\nabla}_{\xi'} \xi' = \nabla_{\xi'} \xi' = 0,$$

which proves the assertion of the proposition.

Corollary 1. *There exists a local section $\xi' \in \Gamma(TM^\perp)$ such that $\tau(\xi') = 0$.*

By using Theorems 2 and 6 and Proposition 2 we obtain

Corollary 2. *Let M be a degenerate hypersurface of $\bar{M}(c)$ such that both fundamental forms B and C vanish identically on M . Then M is locally a product of a semi-Riemannian manifold of constant curvature c (leaf of the screen distribution) and a null geodesic of $\bar{M}$.*

In particular, if $c=0$ in Corollary 2, then based on the fact that geodesics of R_v^{m+1} are straight lines, we derive that M is a ruled hypersurface of R_v^{m+1} .

Next, we note that in case $A_\xi X = \rho X$ for any $X \in \Gamma(TM)$, then since $A_\xi \xi = 0$ we get $A_\xi = 0$. Hence by Theorem 6, M is totally geodesic. That is why we say that M is *umbilical* with respect to N if we have

$$(4.1) \quad A_\xi hX = \rho hX, \quad \forall X \in \Gamma(TM),$$

where ρ is a differentiable function. By using (2.13) we see that (4.1) is equivalent with

$$(4.2) \quad B(hX, hY) = \rho g(hX, hY), \quad \forall X, Y \in \Gamma(TM).$$

Remark 4. According to (2.4), (2.8) and (4.2) we see that in case M is umbilical with respect to one affine normal then it is umbilical with respect

to any other affine normal. Thus we have the motivation to call M satisfying (4.1) and/or (4.2) a *totally umbilical degenerate hypersurface*.

Theorem 7. *Let M be a totally umbilical degenerate hypersurface of $\bar{M}(c)$. Then ρ satisfies the partial differential equation*

$$(4.3) \quad \xi(\rho) + \rho\tau(\xi) - \rho^2 = 0,$$

and the curvature tensor of M is given by

$$(4.4) \quad \begin{aligned} R(X, Y)Z &= g(X, Z)(cY - \rho A_N Y) - g(Y, Z)(cX - \rho A_N X), \\ \forall X, Y, Z &\in \Gamma(TM). \end{aligned}$$

Moreover, if $\dim M > 2$ then ρ satisfies the partial differential equations

$$(4.5) \quad hX(\rho) + \rho\tau(hX) = 0, \quad \forall X \in \Gamma(TM).$$

Proof. Taking account of (3.6) and (4.2) in (2.19) we obtain

$$(4.6) \quad \begin{aligned} X(\rho)g(Y, Z) - Y(\rho)g(X, Z) + \rho\{(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) + \\ + \tau(X)g(Y, Z) - \tau(Y)g(X, Z)\} = 0, \quad \forall X, Y, Z \in \Gamma(TM). \end{aligned}$$

Take $X \in \xi$ and $Z = Y \in \Gamma(SM)$ such that $g(Y, Y) = 0$ and by using (2.5) we obtain

$$(4.7) \quad (\xi(\rho) + \rho\tau(\xi))g(Y, Y) + \rho g(\nabla_Y \xi, Y) = 0.$$

On the other hand, by using (3.9) and (4.1) we infer

$$(4.8) \quad g(\nabla_Y \xi, Y) = \rho g(Y, Y).$$

Thus (4.3) follows from (4.7) taking account of (4.8). Further, (4.4) follows from (2.16) by using (4.2), (2.14) and (3.6). Finally, take $X = hX$, $Y = hY$ and $Z = hZ$ in (4.6) and by using (2.5) obtain

$$(4.9) \quad (hX(\rho) + \rho\tau(hX))hY = (hY(\rho) + \rho\tau(hY))hX,$$

since g is non-degenerate on $\Gamma(SM)$. Suppose now there exists a vector field X_0 such that $hX_0(\rho) + \rho\tau(hX_0) \neq 0$. Then from (4.9) it follows that all vector fields from $\Gamma(SM)$ are colinear with hX_0 . This is not possible in case $\dim M > 2$. Hence in this case (4.5) holds and the proof is complete.

With respect to the existence of totally umbilical degenerate hypersurfaces we prove

Proposition 3. *The lightlike cone M of R_v^{m+1} is a totally umbilical degenerate hypersurface.*

Proof. By using (2.1) and (2.14) and taking into account that ξ is just the position vector field we obtain

$$(4.10) \quad \nabla_X \xi = \bar{\nabla}_X \xi = X, \quad \forall X \in \Gamma(TM).$$

Then by using (1.2) and (2.10) in (4.10) we get

$$(4.11) \quad -A_\xi X - \tau(X)\xi = hX + \eta(X)\xi.$$

Since A_ξ is $\Gamma(SM)$ -valued we obtain $A_\xi hX = -hX$ and $\tau(X) = -\eta(X)$. Hence M is totally umbilical with $\rho = -1$.

Next, we say that the screen distribution SM is *totally umbilical* if we have

$$(4.12) \quad C(X, hY) = \lambda g(X, hY), \quad \forall X, Y \in \Gamma(TM),$$

where λ is a differentiable function on M . From (4.12) we see that C is now symmetric on $\Gamma(SM)$ and hence according to Theorem 1, SM is an integrable distribution. Then (4.12) says that all leaves of SM are totally umbilical immersed in M with respect to the normal $\xi \in \Gamma(TM^\perp)$. This is a motivation of the name *totally umbilical* given to SM . In case $\lambda=0$, each leaf of SM is totally geodesic immersed in M . That is why we say that SM is *proper totally umbilical* when it is totally umbilical and $\lambda \neq 0$.

By using (2.12) and taking into account that SM is non-degenerate with respect to g we see that (4.12) is equivalent with

$$(4.13) \quad A_N X = \lambda hX, \quad \forall X \in \Gamma(TM).$$

As a consequence of (4.12) and (4.13) we obtain

$$(4.14) \quad C(\xi, hY) = 0; \quad A_N \xi = 0.$$

Theorem 8. *Let M be a degenerate hypersurface of $\bar{M}(c)$ such that SM is totally umbilical. Then we have*

(i) *If $\lambda=0$ then $c=0$ and the immersion of M is affinely equivalent to the graph immersion of a certain function $F:M \rightarrow R$;*

(ii) *If $\lambda \neq 0$ then M is totally umbilical immersed in $\bar{M}(c)$.*

Proof. By using (2.5), (2.6), (3.6) and (4.12) in (2.21) we obtain

$$(4.15) \quad \begin{aligned} & \{X(\lambda) - \lambda\tau(X) + c\eta(X)\}g(Y, hZ) - \{Y(\lambda) - \lambda\tau(Y) + c\eta(Y)\} \\ & g(X, hZ) = \lambda\{B(Y, hZ)\eta(X) - B(X, hZ)\eta(Y)\}, \end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$. Take now $X = \xi$ and taking account of (2.6) and (2.14) in (4.15) we derive

$$(4.16) \quad \{\xi(\lambda) - \lambda\tau(\xi) + c\}g(Y, hZ) = \lambda B(Y, hZ).$$

In case $\lambda=0$ we get $c=0$ from (4.16), since $\dim M \geq 2$ and g is non-degenerate on $\Gamma(SM)$. Hence M is a degenerate hypersurface of R^{m+1}_ν with parallel screen distribution. Then according to Theorem 4 we have the assertion (i). The assertion (ii) follows from (4.16).

Corollary 3. *Each degenerate hypersurface M of $\bar{M}(c)$ with proper totally umbilical screen distribution is totally-umbilical immersed in $\bar{M}(c)$.*

As a final remark we may say that in a study of differential geometry of a degenerate hypersurface we may obtain results on the extrinsic geometry of M by using some hypothesis on its intrinsic geometry and vice versa. Thus, in Corollary 3 we have the hypothesis on leaves of the screen distribution (which are pieces of M) and obtain a result with respect to the immersion of M in $\bar{M}(c)$. Similar situations can be observed in Theorems 1, 4, 6, 8. This is a characteristic of the study of degenerate hypersurfaces which makes it very different from that one of non-degenerate hypersurfaces.

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REFERENCES

1. Bonnor W.B., *Null Hypersurfaces in Minkowski Space Time*. Tensor N. S., **24**, 329—345 (1972).
2. Cagnac Fr., *Géométrie de Hypersurfaces Isotropes*. C. R. Acad. Sci. Paris, **201**, 3 045—3 048 (1965).
3. Chen B. Y., *Geometry of Submanifolds*. Marcel Dekker, New York, 1973.
4. Dajczer M., *Submanifolds and Isometric Immersions*. Publish or Perish, Houston, Texas, 1990.
5. Kammerer J. B., *Théorème de Peeling et Hypersurfaces Isotropes*. Rend. del Circolo Mat. di Palermo, Ser. II, **16**, 129—202 (1967).
6. Nomizu K., Pinkall U., *On the Geometry of Affine Immersions*. Math. Z., **195**, 165—178 (1981).
7. O'Neill B., *Semi-Riemannian Geometry with Applications to Relativity*. Academic Press, 1983.
8. Roșca R., *Sur les Hypersurfaces Isotropes de Défaut 1 Inclues dans une Variété Lorentzienne*. C.R. Acad. Sci., Paris, **272**, 393—396 (1979).
9. Roșca R., *On Null Hypersurfaces of a Lorentzian Manifold*. Tensor N. S., **23**, 66—74 (1972).

HIPERSUPRAFETE DEGENERATE ALE VARIETĂȚILOR SEMI-RIEMANNIENE

(Rezumat)

Se dezvoltă o metodă de studiu a geometriei diferențiale a hipersuprafețelor pe care metrica degenerează. Rezultatele obținute se bazează pe noțiunile de distribuție tip ecran și pe conexiunea liniară indusă. Este remarcabil faptul că orice con luminos este o hipersuprafață degenerată total ombilicală în R_0^{m+1} .

CONNEXIONS LINÉAIRES SUR DES ESPACES RIEMANN PARTICULIERS

PAR

ELENA PUȘCAȘU

1. Introduction. Soit V_4 la variété espace-temps de la relativité générale, $g_{\alpha\beta}$ ($\alpha, \beta, \gamma, \dots = 0, 1, 2, 3$), le tenseur métrique de la variété et $p_\alpha = (p_{ab}^a)$, ($a, b, c, \dots = 1, 2, 3, 4$) un système de matrices Dirac. p_α étant matrices Dirac, nous avons

$$(1) \quad p_\alpha p_\beta + p_\beta p_\alpha = -2g_{\alpha\beta} e,$$

e — la matrice unité d'ordre 4.

Les matrices Dirac p_α définissent un spin-tenseur (spineur de type (1, 1), tenseur de type (0, 1)) nommé le *spin-tenseur fondamental* de la variété V_4 .

Si ω est 1-forme de la connexion riemannienne sur V_4 et σ est 1-forme de la connexion spinorielle, la liaison entre les deux connexions est donnée par

$$(2) \quad \sigma_{bp}^a = -\frac{1}{4} \omega_{bp}^\alpha p_{ac}^a p_{\beta}^{\beta c}, \quad \text{où} \quad p^\beta = g^{\beta\alpha} p_\alpha.$$

Soit $\bar{\nabla}$ le symbole de la dérivée covariante par rapport au couple des connexions (σ, ω). Nous avons

$$(3) \quad \bar{\nabla}_\alpha p_{\beta b}^a = 0.$$

Pour des détails voir [1].

2. Les matrices Dirac h_α . En suite, nous supposons que sur V_4 nous avons une 3- Π -structure définie par le tenseur φ_β^α , donc

$$(4) \quad \varphi_\beta^\alpha \varphi_\delta^\beta \varphi_\epsilon^\delta = \lambda^3 \delta_\epsilon^\alpha, \quad \lambda \neq 0.$$

À l'aide des matrices Dirac p_α et du tenseur φ nous définissons les matrices h_β données par

$$(5) \quad h_\beta = \varphi_\beta^\alpha p_\alpha.$$

Tenant compte de (1), (4) et (5) on obtient le

Théorème 1. Une condition nécessaire et suffisante pour que les matrices h_β forment un système de matrices Dirac est que nous avons

$$(6) \quad \lambda^3 \varphi_\alpha^\tau g_{\beta\tau} = \varphi_\beta^\alpha g_{\alpha\tau}, \quad \text{où} \quad \varphi_\beta^\alpha = \varphi_\beta^\alpha \varphi_\alpha^\tau.$$

Étant donnée la connexion spinorielle σ , nous proposons, en suite, de déterminer la connexion linéaire Γ ainsi que la dérivée covariante du tenseur $h_{\beta b}^a$ par rapport au couple de connexions (σ, Γ) soit nulle.

Si ∇ est le symbole de la dérivation covariante par rapport au couple (σ, Γ) , nous avons

$$(7) \quad \nabla_{\alpha} h_{\beta b}^a = \partial_{\alpha} h_{\beta b}^a + \sigma_{\alpha}^a h_{\beta b}^c - \sigma_{b\alpha}^c h_{\beta c}^a - \Gamma_{\beta\alpha}^{\tau} h_{\tau b}^a = 0.$$

De (2), (5) et (7) résulte

$$(8) \quad \Gamma_{\alpha\beta}^{\tau} = \omega_{\alpha\beta}^{\tau} + \frac{1}{\lambda^3} \varphi_{\rho}^{(2)} \nabla_{\beta} \varphi_{\alpha}^{\rho},$$

où ∇ est le symbole de la dérivation covariante par rapport à la connexion ω .

3. Certaines propriétés de la connexion Γ .

(P₁). Soit T le tenseur de torsion de la connexion Γ est N le tenseur de Nijenhuis de la 3-II-structure φ . Le tenseur N est donné par

$$(9) \quad N_{\beta\gamma}^{\alpha} = \varphi_{\beta}^{(2)} (\partial_{\tau} \varphi_{\gamma}^{\alpha} - \partial_{\gamma} \varphi_{\tau}^{\alpha}) - \varphi_{\gamma}^{(2)} (\partial_{\tau} \varphi_{\beta}^{\alpha} - \partial_{\beta} \varphi_{\tau}^{\alpha}).$$

De (8) et (9), nous avons

$$(10) \quad N_{\beta\gamma}^{\alpha} = \varphi_{\beta}^{(2)} T_{\gamma\tau}^{\tau} \varphi_{\tau}^{\alpha}.$$

Il en résulte le

Théorème 2. Une condition nécessaire et suffisante pour que la 3-II-structure φ soit intégrable est que le tenseur de torsion de la connexion Γ vérifie la relation

$$(11) \quad \varphi_{\beta}^{(2)} T_{\gamma\tau}^{\tau} \varphi_{\tau}^{\alpha} = 0.$$

(P₂). Le couple des connexions (ω, Γ) forment un couple de connexions compatibles avec la 3-II-structure φ .

Vraiment, si le vecteur v^{α} se transporte par parallélisme au long d'une certaine courbe de V_4 par rapport à la connexion Γ , le vecteur $u^{\alpha} = \varphi_{\beta}^{\alpha} v^{\beta}$ se transporte par parallélisme par rapport à la connexion ω sur la même courbe.

(P₃). Soit: L — le module des champs de vecteur sur V_4 ; ρ — l'opérateur de courbure mixte de la paire (ω, Γ) ; r — l'opérateur de courbure de la connexion ω ; R — l'opérateur de courbure de la connexion Γ ; T — l'opérateur de torsion de la connexion Γ .

Maintenant, nous allons introduire la notion d'algèbre de déformation de couple (ω, Γ) (v. [2], [4]).

Considérons l'opérateur de déformation de la paire (ω, L)

$$(12) \quad \tau_X = \nabla_X - \overset{\circ}{\nabla}_X, \quad \forall X \in L.$$

À l'aide de l'opérateur τ_X on peut définir une opération entre les champs de vecteurs de L donnée par

$$(13) \quad X * Y = \tau_X Y, \quad \forall X, Y \in L.$$

On vérifie aisément que, par rapport à cette opération, L est une algèbre. L'algèbre définie par (13) sera nommée algèbre de déformation de la paire (ω, Γ) et sera notée par A .

Enfin, on peut introduire l'opérateur

$$(14) \quad K(X, Y) = \frac{1}{4}[\tau_X, \tau_Y].$$

Compte tenu de (12) et (14) on trouve

$$(15) \quad R(X, Y) + r(X, Y) = 2\rho(X, Y) + 4K(X, Y).$$

Un calcul simple nous conduit aux résultats suivants :

$$(16) \quad X * Y - Y * X = T(X, Y),$$

$$(17) \quad X * Y + Y * X = \tau_X Y + \tau_Y X = T(X, Y) + 2\tau_Y X,$$

$$(18) \quad X * (Y * Z) - Y * (X * Z) = 4K(X, Y)Z.$$

De ces formules il résulte :

a) L'algèbre A est commutative si et seulement si la connexion Γ est sans torsion.

b) L'algèbre A est anti commutative si et seulement si

$$(19) \quad T(X, Y) = 2\tau_X Y.$$

c) Dans l'algèbre A on a la loi

$$(20) \quad X * (Y * Z) = Y * (X * Z),$$

si et seulement si $K(X, Y)Z = 0$, c'est-à-dire, d'après (15) la courbure mixte de la paire (ω, Γ) est la moyenne arithmétique des courbures de ω et Γ .

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BIBLIOGRAPHIE

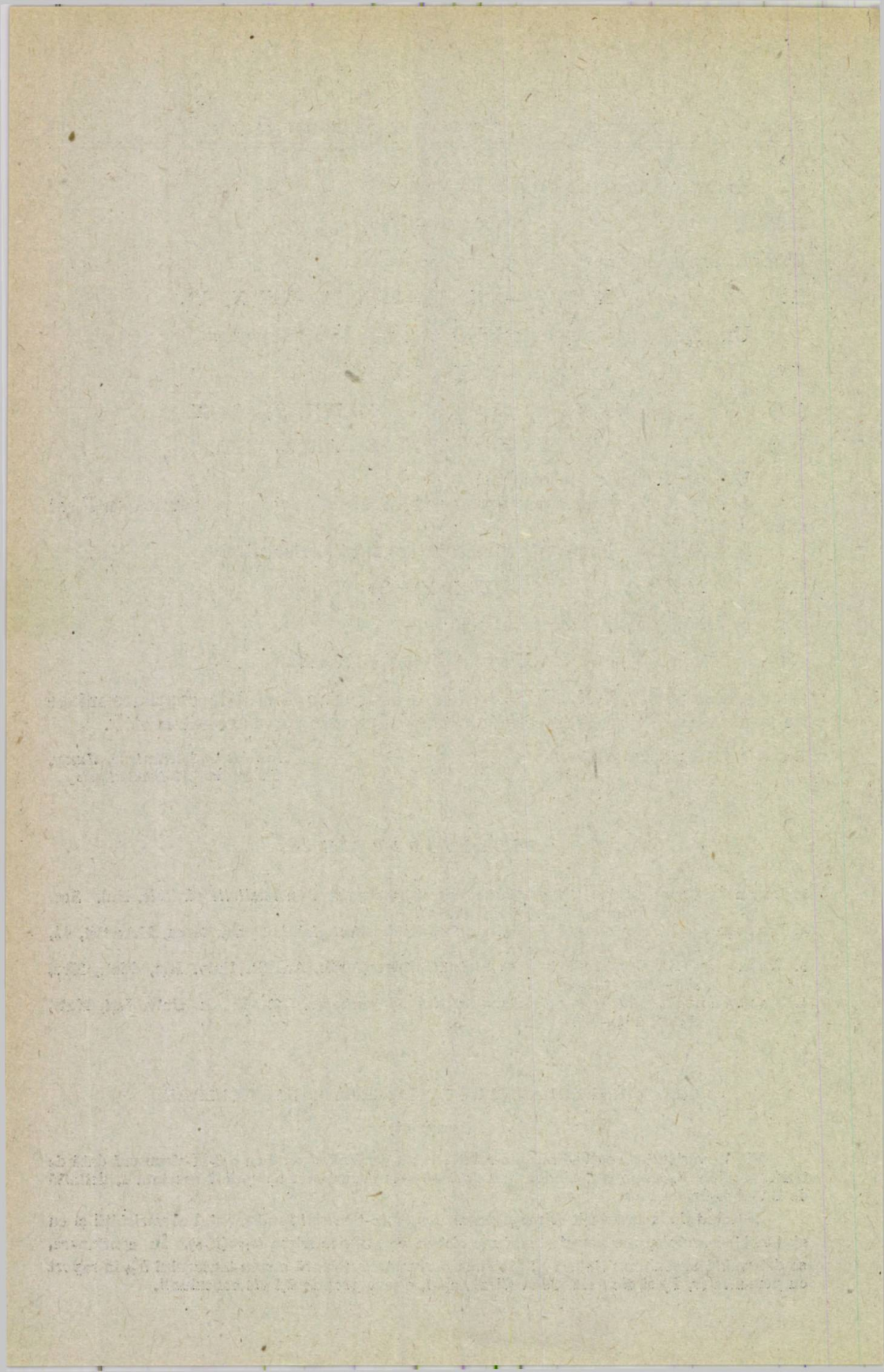
1. Lichnerowicz A., *Champs spinoriels et propagateurs en relativité générale*. Bul. Soc. Math. France, **92**, 11—100 (1964).
2. Vaisman I., *Sur quelques formules du calcul de Ricci global*. Comm. Math. Helvetici, **41**, 2, 78—87 (1966—1967).
3. Vamanu E., *Quasi-conexiuni pe varietăți diferențiabile*. An. Șt. Univ. Iași, Mat., **XVI**, 383—388 (1970).
4. Vamanu E., *Quasi-conexiuni sur variétés différentiables (II)*. An. Șt. Univ. Iași, Mat., **XVIII**, 161—166 (1972).

CONEXIUNI LINIARE PE SPAȚII RIEMANN PARTICULARE

(Rezumat)

Fie V_4 varietatea spațiu-timp din relativitatea generală dotată cu o 3-II-structură dată de tensorul φ . Pe V_4 avem conexiunea riemanniană ω și conexiunea spinorială asociată σ , definită de Lichnerowicz.

Plecînd de la matricele Dirac p_α care dau spin-tensorul fundamental al varietății și cu ajutorul tensorului φ se construiește un nou sistem de matrice Dirac $h_\alpha = (h_{\alpha b}^a)$. În continuare, se determină conexiunea liniară Γ , astfel ca derivata covariantă a spin-tensorului $h_{\alpha b}^a$ în raport cu perechea (σ, Γ) să fie nulă. Se studiază, apoi, câteva proprietăți ale conexiunii.



ANOTHER PSEUDO-RIEMANNIAN STRUCTURE ON THE COTANGENT BUNDLE

BY

V. ÔPROIU and N. PAPAGHIUC

1. Introduction. Let M be a smooth manifold and T^*M its cotangent bundle. In [3] the authors have studied the properties of a pseudo-Riemannian metric G on T^*M , similar to a Riemann extension (see [4]). This metric was obtained by using an arbitrary symmetric non linear connection on T^*M . In this paper we study the properties of another pseudo-Riemannian metric $\tilde{G}$ on T^*M obtained from G by adding a term defined by a symmetric M -tensor field of type $(0,2)$ on T^*M . This metric can be considered as a metric of the same type as G by using another symmetric nonlinear connection on T^*M . However we studied the properties of this metric by using the adapted local frames on T^*M defined by the initial symmetric nonlinear connection. We find the Levi-Civita connection of the pseudo-Riemannian metric $\tilde{G}$ on T^*M , the curvature tensor field and we study the conditions under which the considered pseudo-Riemannian metric $\tilde{G}$ is flat or conformally flat.

2. The Cotangent Bundle T^*M . Throughout this paper the indices h, i, j, k, l run over the range $\{1, \dots, n\}$. Summation over repeated indices is always implied.

Let M be an n -dimensional smooth (i.e. C^∞) real manifold and denote by $\pi: T^*M \rightarrow M$ the cotangent bundle of M . Then T^*M is a $2n$ -dimensional smooth manifold and some local charts induced from local charts of M may be used. Let (U, x) be a local chart in M with coordinate neighbourhood U and coordinate function $x=[x^i]$. Then the local chart $(\pi^{-1}(U), (q, p))$ is induced in T^*M with coordinate neighbourhood $\pi^{-1}(U)$ and coordinate function $(q, p)=[q^i, p_i]$ where $q^i=x^i \cdot \pi$ and p_i are vector space coordinates of elements from $\pi^{-1}(U)$ with respect to the local natural frame $(dx^1, \dots, dx^n)$ of T^*M defined over U by (U, x) .

Consider the tangent bundle TT^*M of T^*M and let $VT^*M = \text{Ker } \pi_*$ be the vertical distribution over T^*M . Then VT^*M is involutive with fibre dimension n and if $(\pi^{-1}(U), (q, p))$ is an induced local chart on T^*M then the vector fields $E^i = \partial/\partial p_i$; $i=1, \dots, n$ define a frame in VT^*M over $\pi^{-1}(U)$. A complementary distribution HT^*M to VT^*M in TT^*M (a horizontal distribution) i. e.

(1)

$$TT^*M = VT^*M \oplus HT^*M,$$

defines a nonlinear connection on T^*M . If $(\pi^{-1}(U), (q, p))$ is an induced local chart on T^*M then a (nonholonomic) frame $(E_1, \dots, E_n)$ may be assigned to HT^*M over $\pi^{-1}(U)$, where

$$E_i = \frac{\partial}{\partial q^i} - N_{ij} E^j.$$

The coefficients N_{ij} are the coefficients of the nonlinear connection defined by HT^*M with respect to the induced local chart $(\pi^{-1}(U), (q, p))$. Considering the change rule for the coefficients N_{ij} when a change of induced local charts on T^*M is performed it follows that the symmetry condition $N_{ij} = N_{ji}$ is meaningful. A nonlinear connection on T^*M such that the coefficients N_{ij} with respect to induced local charts on T^*M are symmetric is called a symmetric nonlinear connection on T^*M . A symmetric nonlinear connection on T^*M there always exists. From now on we shall assume that the nonlinear connection we use is symmetric.

An M -tensor field of type (k, l) on T^*M is defined by sets of local coordinate components $T_{j_1 \dots j_l}^{i_1 \dots i_k}$ assigned to every induced local chart $(\pi^{-1}(U), (q, p))$ on T^*M , obeying the change rule of the local coordinate components of a tensor field of type (k, l) on M . A linear M -connection on T^*M may be defined similarly by the connection coefficients $\Gamma_{jk}^i(q, p)$ assigned to every induced local chart on T^*M obeying the change rule of the connection of a linear connection on M (see [1] for details).

Remark. A tensor field on M can be thought of as an M -tensor field of the same type on T^*M with the components independent of the cotangential coordinates. A linear connection on M can be thought of as a linear M -connection on T^*M .

The torsion of the nonlinear connection on T^*M is defined by the local coordinate components R_{ijk} assigned to the induced local chart $(\pi^{-1}(U), (q, p))$ by

$$(2) \quad R_{ijk} = \frac{\delta N_{ki}}{\delta q^j} - \frac{\delta N_{ji}}{\delta q^k},$$

where

$$\frac{\delta N_{ki}}{\delta q^j} = E_j N_{ki} = \frac{\partial N_{ki}}{\partial q^j} - N_{jn} \frac{\partial N_{ki}}{\partial p_n}.$$

It follows that the components R_{ijk} define an M -tensor field of type $(0, 3)$ on T^*M . Due to the formula

$$(3) \quad [E_i, E_j] = -R_{kij} E^k,$$

it follows that the distribution HT^*M is involutive if and only if the torsion of the corresponding nonlinear connection is zero. From the symmetry of the considered nonlinear connection we get the identity

$$(4) \quad \sum_{(i,j,k)} R_{ijk} = 0,$$

where $\sum_{(i,j,k)}$ denotes the cyclic sum with respect to (i, j, k) .

We have

$$(5) \quad [E^i, E_j] = \Phi_{jk}^i E^k,$$

and the components $\Phi_{jk}^i = -\partial N_{jk}/\partial p_i$ define a linear M -connection on T^*M . This linear M -connection may be used to define a differential operator $\bar{\nabla}$ acting on the M -tensor fields on T^*M . The local coordinate expression of $\bar{\nabla}$ is similar to that of the covariant differential defined by an usual linear connection on M . E. g., from an M -tensor field of type $(1, 1)$ with the components $T_j^i(q, p)$ we get

$$(6) \quad \bar{\nabla}_h T_j^i = \frac{\delta T_j^i}{\delta q^h} + \Phi_{hl}^i T_j^l - \Phi_{hj}^l T_l^i,$$

which define an M -tensor field of type $(1, 2)$.

Then from R_{ijk} we get

$$(6') \quad \bar{\nabla}_h R_{ijk} = \frac{\delta R_{ijk}}{\delta q^h} - \Phi_{hl}^i R_{ljk} - \Phi_{hj}^l R_{ilk} - \Phi_{hk}^l R_{ijl}.$$

3. The Pseudo-Riemannian Metric on T^*M . By using a symmetric non linear connection on T^*M we may consider a pseudo-Riemannian metric G on T^*M , of Riemann extension type

$$(7) \quad G = 2\delta p_i dq^i,$$

where

$$(8) \quad \delta p_i = dp_i + N_{ij} dq^j.$$

Let C be a symmetric M -tensor field of type $(0, 2)$ on T^*M defined by the local coordinate components $c_{ij}(q, p)$. Then we may consider another pseudo-Riemannian metric $\tilde{G}$ on T^*M , defined by

$$(9) \quad \tilde{G} = 2\delta p_i dq^i + c_{ij} dq^i dq^j.$$

Remark that the distribution VT^*M is isotropic with respect to $\tilde{G}$. Moreover, since we may write

$$\tilde{G} = 2(\delta p_i + \frac{1}{2} c_{ij} dq^j) dq^i,$$

it follows that the signature of $\tilde{G}$ is (n, n) and $\tilde{G}$ may be thought of as a pseudo-Riemannian metric of Riemann extension type defined by the nonlinear symmetric connection with the coefficients

$$(10) \quad \tilde{N}_{ij} = N_{ij} + \frac{1}{2} c_{ij}.$$

Denote by $\tilde{H}T^*M$ the horizontal distribution on T^*M defined by this new non linear connection. The frame $(\tilde{E}_1, \dots, \tilde{E}_n)$ assigned to $\tilde{H}T^*M$ by the induced local chart $(\pi^{-1}(U), (q, p))$ is related to the corresponding local frame of HT^*M by

$$(11) \quad \tilde{E}_i = E_i - \frac{1}{2} c_{ij} E^j,$$

and

$$(11') \quad \tilde{\delta} p_i = \delta p_i + \frac{1}{2} c_{ij} dq^j.$$

We have also the following formulae relating the corresponding linear M -connections and the torsions

$$(12) \quad \tilde{\Phi}_{ij}^k = \Phi_{ij}^k - D_{ij}^k,$$

$$(13) \quad \tilde{R}_{ijk} = R_{ijk} + \frac{1}{2} \{ \bar{\nabla}_j c_{ki} - \bar{\nabla}_k c_{ji} - c_{jh} D_{ki}^h + c_{kh} D_{ji}^h \},$$

where

$$D_{ij}^k = \frac{1}{2} \frac{\partial c_{ij}}{\partial p_k}.$$

The Levi-Civita connection ∇ of the pseudo-Riemannian metric G has the following local coordinate expression (see [3])

$$(14) \quad \nabla^i E^j = 0, \quad \nabla_i E^j = -\Phi_{ik}^j E^k, \quad \nabla^i E_j = 0, \quad \nabla_i E_j = \Phi_{ij}^k E_k + R_{ij,k} E^k,$$

where

$$\nabla^i = \nabla_{E^i}, \quad \nabla_i = \nabla_{E_i}.$$

The Schouten connection $\bar{\nabla}$ defined by ∇ and the decomposition (1) of TT^*M is given by

$$(15) \quad \bar{\nabla}^i E^j = 0, \quad \bar{\nabla}_i E^j = -\Phi_{ik}^j E^k, \quad \bar{\nabla}^i E_j = 0, \quad \bar{\nabla}_i E_j = \Phi_{ij}^k E_k.$$

Remark. Consider an M -tensor field on T^*M of type (let us say) $(1,1)$ defined by the components $T_j^i(q, p)$ in the induced local chart $(\pi^{-1}(U), (q, p))$. Then the following tensor fields may be considered on T^*M :

$$T = T_j^i E_i \otimes E^j, \quad T' = T_j^i E_i \otimes dq^j, \quad T'' = T_j^i \delta p_i \otimes E^j, \quad T''' = T_j^i \delta p_i \otimes dq^j.$$

We may consider the covariant differentiation of these tensor fields with respect to the Schouten connection $\bar{\nabla}$. It is a remarkable property of the Levi-Civita connection ∇ that the components of the tensor fields $\bar{\nabla} T$, $\bar{\nabla} T'$, $\bar{\nabla} T''$, $\bar{\nabla} T'''$ are the same, namely $\bar{\nabla}^h T_j^i = \partial T_j^i / \partial p_h$ and $\bar{\nabla}_h T_j^i$ defined by (6). This is the explanation of denoting by the same symbol $\bar{\nabla}$ the Schouten connection and the differential operator defined by (6).

Denote by $\tilde{\nabla}$ the Levi-Civita connection of the pseudo-Riemannian metric $\tilde{G}$ defined by (9) on T^*M . Then its expression with respect to the local frame $(E^1, \dots, E^n, \tilde{E}_1, \dots, \tilde{E}_n)$ is given by formulae similar to (14) where ∇ is replaced by $\tilde{\nabla}$, E_i by $\tilde{E}_i$, Φ_{ij}^k by $\tilde{\Phi}_{ij}^k$, R_{ijk} by $\tilde{R}_{ijk}$. Using the formulae (10), (11), (12), (13) we get

Proposition 1. *The expression of $\tilde{\nabla}$ with respect to the local frame (E^i, E_i) on T^*M is given by*

$$(16) \quad \tilde{\nabla}^i E^j = 0, \quad \tilde{\nabla}_i E^j = -\tilde{\Phi}_{ih}^j E^k, \quad \tilde{\nabla}^i E_j = D_{jh}^i E^k, \quad \tilde{\nabla}_i E_j = \tilde{\Phi}_{ij}^h E_k + A_{ijk} E^k,$$

where

$$(17) \quad \begin{aligned} A_{ijk} &= R_{ijk} + \frac{1}{2} (\bar{\nabla}_i c_{jk} + \bar{\nabla}_j c_{ik} - \bar{\nabla}_k c_{ij}) + c_{kh} D_{ij}^h = \\ &= \tilde{R}_{ijk} + \frac{1}{2} (\bar{\nabla}_i c_{jk} + c_{jh} D_{ik}^h + c_{kh} D_{ij}^h). \end{aligned}$$

Remark. Assume that the components c_{ij} are independent of p . Then $D_{ij}^h = 0$ and the formulae (16) become simpler

$$(16') \quad \tilde{\nabla}^i E^j = 0, \quad \tilde{\nabla}_i E^j = -\tilde{\Phi}_{ih}^j E^k, \quad \tilde{\nabla}^i E_j = 0, \quad \tilde{\nabla}_i E_j = \Phi_{ij}^k E_k + A_{ijk} E^k,$$

where

$$(17'') \quad A_{ijk} = R_{ijk} + \frac{1}{2} (\bar{\nabla}_i c_{jk} + \bar{\nabla}_j c_{ik} - \bar{\nabla}_k c_{ij}) = \tilde{R}_{ijk} + \frac{1}{2} \bar{\nabla}_i c_{jk}.$$

Assume further that the chosen nonlinear connection on T^*M is obtained from linear connection $\tilde{\nabla}$ on M i. e.

$$N_{ij} = -\Gamma_{ij}^k p_k,$$

where Γ_{ij}^k are the connection coefficients of $\tilde{\nabla}$. Then

$$(16'') \quad \tilde{\nabla}^i E^j = 0, \quad \tilde{\nabla}_i E^j = -\Gamma_{ih}^j E^k, \quad \tilde{\nabla}^i E_j = 0, \quad \tilde{\nabla}_i E_j = \Gamma_{ij}^k E_k + A_{ijk} E^k,$$

where

$$(17''') \quad A_{ijk} = -p_h R_{ijk}^h + \frac{1}{2} (\tilde{\nabla}_i c_{jk} + \tilde{\nabla}_j c_{ik} - \tilde{\nabla}_k c_{ij}).$$

In the last formula R_{ijk}^h are the components of the curvature tensor field R of the linear connection $\tilde{\nabla}$ on M and $\tilde{\nabla}_i c_{jk}$ are the components of the covariant derivative of C with respect to $\tilde{\nabla}$.

4. The Curvature Tensor Field. Denote by K the curvature tensor field of ∇ . Then its expression with respect to the local frame (E^i, E_i) on T^*M is given by

$$(18) \quad \begin{aligned} K(E^i, E^j)E^k &= K(E^i, E^j)E_k = 0, \quad K(E^i, E_j)E^k = -K_{jh}^{ik} E^h, \\ K(E_i, E_j)E^k &= -K_{hij}^k E^h, \quad K(E^i, E_j)E_k = K_{jk}^{ih} E_h - K_{jkh}^i E^h, \\ K(E_i, E_j)E_k &= K_{kij}^h E_h + K_{hki j} E^h, \end{aligned}$$

where

$$(19) \quad \begin{aligned} K_{jk}^{ih} &= \frac{\partial \Phi_{jh}^i}{\partial p_h} = -\frac{\partial^2 N_{jk}}{\partial p_i \partial p_h}, \\ K_{kij}^h &= -\frac{\partial R_{kij}}{\partial p_h} = \frac{\delta \Phi_{kj}^h}{\delta q^i} - \frac{\delta \Phi_{ki}^h}{\delta q^j} + \Phi_{it}^h \Phi_{jk}^t - \Phi_{jt}^h \Phi_{ik}^t, \\ K_{hki j} &= -\bar{\nabla}_i R_{jh k} + \bar{\nabla}_j R_{ih k}. \end{aligned}$$

Denote by $\tilde{K}$ the curvature tensor field of the connection $\tilde{\nabla}$ on T^*M . Then $\tilde{K}$ may be expressed just like K by formulae similar to (18), (19) with respect to the local frame $(E^i, \tilde{E}_i)$. However, it is convenient for us to get the expression of $\tilde{K}$ with respect to the initial local frame (E^i, E_i) .

Proposition 2. *The expression of $\tilde{K}$ with respect to the local frame (E^i, E_i) of T^*M is given by*

$$(20) \quad \begin{aligned} \tilde{K}(E^i, E^j)E^k &= \tilde{K}(E^i, E^j)E_k = 0, \quad \tilde{K}(E^i, E_j)E^k = -\tilde{K}_{jh}^{ik}E^h, \\ \tilde{K}(E_i, E_j)E^k &= -\tilde{K}_{hij}^kE^h, \quad \tilde{K}(E^i, E_j)E_k = \tilde{K}_{jk}^{ih}E_h - \tilde{K}_{jkh}^iE^h, \\ \tilde{K}(E_i, E_j)E_k &= \tilde{K}_{kij}^hE_h + \tilde{K}_{hki j}E^h, \end{aligned}$$

where

$$\tilde{K}_{jh}^{ik} = -\frac{\partial^2 \tilde{N}_{jk}}{\partial p_i \partial p_h} = K_{jk}^{ih} - \frac{\partial D_{jk}^h}{\partial p_i},$$

$$(21) \quad \begin{aligned} \tilde{K}_{kij}^h &= \frac{\delta \tilde{\Phi}_{kj}^h}{\delta q^i} - \frac{\delta \tilde{\Phi}_{ki}^h}{\delta q^j} + \tilde{\Phi}_{it}^h \tilde{\Phi}_{kj}^t - \tilde{\Phi}_{jt}^h \tilde{\Phi}_{ki}^t = K_{kij}^h - \bar{\nabla}_i D_{jk}^h + \\ &+ \bar{\nabla}_j D_{ik}^h + D_{il}^h D_{jk}^l - D_{jl}^h D_{ik}^l, \\ \tilde{K}_{jkh}^i &= -\frac{\partial A_{jkh}}{\partial p_i} + \bar{\nabla}_j D_{kh}^i + D_{kl}^i D_{jh}^l + D_{hl}^i D_{jk}^l = \tilde{K}_{jk}^{ih} + c_{hi} \tilde{K}_{jk}^l, \end{aligned}$$

$$\tilde{K}_{hki j} = \bar{\nabla}_i A_{jkh} - \bar{\nabla}_j A_{ikh} - A_{ih} D_{jk}^l + A_{jl} D_{ik}^l + A_{jil} D_{kh}^l - A_{ikh} D_{jl}^l + R_{l i j} D_{kh}^l.$$

Next we get by a straightforward computation

Proposition 3. *The local coordinate relations obtained from the first Bianchi identity satisfied by $\tilde{K}$ are*

$$(22) \quad \tilde{K}_{jkh}^i = \tilde{K}_{kjh}^i = \tilde{K}_{hjk}^i, \quad \tilde{K}_{hjk}^i = \tilde{K}_{kjh}^i - \tilde{K}_{jkh}^i, \quad \sum_{(i,j,k)} \tilde{K}_{ijk}^h = 0, \quad \sum_{(i,j,k)} \tilde{K}_{hijk} = 0.$$

The local coordinate relations obtained from the second Bianchi identity satisfied by $\tilde{K}$ are much more complicated so we do not write them down.

5. Flat and Conformally Flat Pseudo-Riemannian Metrics on T^*M . In this section we study the conditions under which the pseudo-Riemannian metric $\tilde{G}$ on T^*M given by (9) is flat or conformally flat. From the Proposition 2 we have that $\tilde{K}=0$ if and only if $\tilde{K}_{jk}^{ih}=0$, $\tilde{K}_{kij}^h=0$, $\tilde{K}_{jkh}^i=0$, $\tilde{K}_{hki j}=0$. Using the first formula (21) and the formula (10) we get

$$(23) \quad \frac{1}{2} c_{jk} = t_{jk} - \Gamma_{jk}^i p_i - N_{jk},$$

where Γ_{jk}^i , t_{jk} are independent of p . Moreover Γ_{jk}^i are the connection coefficients of a torsion free linear connection $\tilde{\nabla}$ on M and t_{jk} are the local coordinated components of a symmetric tensor field of type $(0, 2)$ on M . Next from (12) we find $\tilde{\Phi}_{ij}^k = \Gamma_{ij}^k$, then, using (21) and the condition for $\tilde{\nabla}$ to be flat we find $R_{kij}^h = \tilde{K}_{kij}^h = 0$, where R_{kij}^h are the components of the curvature tensor field of $\tilde{\nabla}$. Finally, after a quite long computation we get

$$(24) \quad \tilde{K}_{hki j} = -\tilde{\nabla}_i(\tilde{\nabla}_h t_{jk} - \tilde{\nabla}_k t_{jh}) + \tilde{\nabla}_j(\tilde{\nabla}_h t_{ik} - \tilde{\nabla}_k t_{ih}) = 0.$$

Thus we may state

Theorem 4. The pseudo-Riemannian metric $\tilde{G}$ given by (9) on T^*M is locally flat if and only if the M -tensor field defined by c_{ij} may be expressed by (23) where the components Γ_{ij}^h , independent of p , define a flat linear connection $\tilde{\nabla}$ on M and the symmetric tensor field of type $(0, 2)$ defined by t_{ij} on M satisfies the condition (24).

Let $R_{jk} = R_{khj}^h$ be the local coordinate components of the Ricci tensor field of a torsion free linear connection on M with the curvature tensor field defined by R_{hij}^h . By studying the condition under which the conformal curvature tensor field of $(T^*M, \tilde{G})$ vanishes we get

Theorem 5. The pseudo-Riemannian space $(T^*M, \tilde{G})$ with $\dim M > 2$ is conformally flat if and only if the M -tensor field defined by c_{jk} may be expressed by (23) where the components Γ_{ij}^h , independent of p , define a projectively flat, torsion free linear connection $\tilde{\nabla}$ on M with symmetric Ricci tensor field R_{ij} and the symmetric tensor field of type $(0, 2)$ defined by t_{ij} on M satisfies the condition

$$(25) \quad \begin{aligned} \tilde{\nabla}_k(\tilde{\nabla}_i t_{jh} - \tilde{\nabla}_j t_{ih}) - \tilde{\nabla}_h(\tilde{\nabla}_i t_{jk} - \tilde{\nabla}_j t_{ik}) = \frac{1}{n-1} (t'_{hi} R_{jk} + \\ + t_{jk} R_{hi} - t_{ki} R_{jh} - t_{jh} R_{ki}). \end{aligned}$$

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REFERENCES

1. Mok K. R., Patterson E. M., Wong Y. C., *Structure of Symmetric Tensors of Type (0,2) and Tensors of Type (1,1) on the Tangent Bundle*. Trans. Am. Math. Soc., **234**, 253–278 (1977).
2. Oproiu V., Papaghiuc N., *The Legendre Transformation and a Pseudo-Riemannian Metric*. An. Șt. Univ. „Al. I. Cuza” Iași, Mat., **35** (1989).
3. Oproiu V., Papaghiuc N., *A Pseudo-Riemannian Structure on the Cotangent Bundle* (to appear).
4. Patterson E. M., Walker A. G., *Riemann Extensions*. Quart. J. Math., (2), **3**, 19–28 (1952).
5. Willmore T. J., *An Introduction to Differential Geometry*. Oxford Univ. Press, Oxford, 1959.
6. Yano K., Ishihara S., *Tangent and Cotangent Bundles*. M. Dekker, New York, 1973.

O ALTĂ STRUCTURĂ PSEUDO-RIEMANNIANĂ PE FIBRATUL COTANGENT

(Rezumat)

În lucrarea [3] sînt studiate proprietățile unei metrici pseudo-riemanniene G pe fibratul cotangent T^*M , asemănătoare cu o extensiune Riemann (v. [4]).

În acest articol se studiază proprietățile unei alte metrici pseudo-riemanniene $\tilde{G}$ pe T^*M obținută din G prin adăugarea unui termen definit de un cîmp M -tensorial simetric de tip $(0,2)$ pe T^*M . Se determină conexiunea Levi-Civita a metricii pseudo-riemanniene $\tilde{G}$ pe T^*M , cîmpul tensorial de curbură, precum și condițiile în care metrica pseudo-riemanniană considerată $\tilde{G}$ pe T^*M este plată sau conform plată.

REGULAR MAPPINGS IN FINSLER SPACES

1. ELLIPSOIDS AND REGULAR MAPPINGS IN VECTORIAL NORMED SPACES

BY

VERONICA T. BORCEA and AL. NEAGU

0. Introduction. The aim of this Note is to determine the characteristic of an affine transformation and of a differentiable homeomorphism between two normed vectorial spaces. This results will be applied to the study of the regular mappings between Finsler spaces.

1. Ellipsoids in Vectorial Normed Spaces. Let V be a vectorial normed space. An ellipsoid in V is the image of a sphere $S(0, t)$ by an affine transformation $A_1: V \rightarrow V$, $Y = A_1X = UX + Y_0$, where $U: V \rightarrow V$ is an isomorphism of vectorial topological spaces, $E_t(U, Y_0) = A_1(S(0, t))$.

We have $E_t(U, Y_0) = \{Y \mid \|U^{-1}(Y - Y_0)\| = t\}$. The operator U is called the characteristic of the family of ellipsoids $\{E_t(U, Y_0) \mid t \in (0, \infty)\}$.

The extreme semi-axes of $E_t(U, Y_0)$ are

$$(1) \quad a_0(t) = \inf_{\|X\|=t} \|A_1X - Y_0\|, \quad a_1(t) = \sup_{\|X\|=t} \|A_1X - Y_0\|.$$

Since U is a linear continuous, bijective operator, it follows that U and U^{-1} are C -isometries, i. e.

$$(2) \quad \frac{1}{C} \|X\| \leq \|UX\| \leq C \|X\|, \quad \frac{1}{C} \|X\| \leq \|U^{-1}X\| \leq C \|X\|,$$

where $C = \max\{\|U\|, \|U^{-1}\|\}$. In addition

$$(3) \quad \inf_{\|X\|=1} \|UX\| = \frac{1}{\|U^{-1}\|}.$$

The formulas (1) become

$$(4) \quad a_0(t) = \frac{t}{\|U^{-1}\|}, \quad a_1(t) = t\|U\|.$$

We denote by $p_1 = a_1(t)/a_0(t)$ and so

$$(5) \quad p_1 = \|U\| \|U^{-1}\|.$$

An ellipsoid $E_t(U, Y_0)$ for which $\|U\| = \|U^{-1}\|$ is called normalized (weakly) and its characteristic U is called normalized characteristic.

If $E_t(U, Y_0)$ is a normalized ellipsoid then by (2) we have

$$(6) \quad \frac{\|X\|}{\|U\|} \leq \|UX\| \leq \|U\| \|X\|$$

and from $I = UU^{-1}$, it follows that $1 \leq \|U\| \|U^{-1}\| = \|U\|^2$, hence $\|U\| \geq 1$.

Proposition 1. Every family of ellipsoids $\{E_t(U, Y_0), t > 0\}$ admits a normalized characteristic.

Proof. Let us consider $U_n = \sqrt{\|U^{-1}\| \|U\|^{-1}} U$, $t_n = t \sqrt{\|U\| \|U^{-1}\|^{-1}}$ and the ellipsoid $E_{t_n}(U_n, Y_0)$ which has the equation $\|U_n^{-1}(Y - Y_0)\| = t_n$.

It follows that $\sqrt{\|U\| \|U^{-1}\|^{-1}} U^{-1}(Y - Y_0) = t \sqrt{\|U\| \|U^{-1}\|^{-1}}$ and hence $\|U^{-1}(Y - Y_0)\| = t$.

We obtain that $E_{t_n}(U_n, Y_0) = E_t(U, Y_0)$ and

$$\|U_n\| = \sqrt{\|U^{-1}\| \|U\|^{-1}} \|U\| = \sqrt{\|U\| \|U^{-1}\|}, \quad \|U_n^{-1}\| = \frac{1}{\sqrt{\|U^{-1}\| \|U\|^{-1}}} \|U^{-1}\| = \sqrt{\|U\| \|U^{-1}\|}$$

so that $\|U_n\| = \|U_n^{-1}\|$.

Proposition 2. The ellipsoid $E_t(U, Y_0)$ is a sphere $S(Y_0, r)$ iff $J = U/\|U\|$ is an isomorphism of vectorial normed spaces (isometric operator) and $r = t\|U\|$.

Proof. $E_t(U, Y_0)$ is a sphere iff $a_1(t) = a_0(t)$ and, by (4), it follows that $t\|U\| = t/\|U^{-1}\| = r$ and so $\|U\| \|U^{-1}\| = 1$. Then $\|U_n\| = \sqrt{\|U^{-1}\| \|U\|^{-1}} \|U\| = 1$ and since U_n is a normalized characteristic, we have $\|X\|/\|U_n\| \leq \|U_n X\| \leq \|U_n\| \|X\|$, hence $\|U_n X\| = \|X\|$. It follows that $U_n = \sqrt{\|U^{-1}\| \|U\|^{-1}} U = (1/\|U\|) U$ is an isomorphism of vectorial normed spaces.

Let us consider the affine transformation $A: V \rightarrow \tilde{V}$, $Y = AY = GY + a$ and $\tilde{Y}_0 = GY_0 + a$, where $G: V \rightarrow \tilde{V}$ is an isomorphism of vectorial topological spaces and a is a fixed point in V .

We suppose that V and $\tilde{V}$ are isomorphic vectorial normed spaces, i. e. there exists $I: \tilde{V} \rightarrow V$ a continuous bijective linear operator such that $\|IX\| = \|\tilde{X}\|$, for every $\tilde{X} \in \tilde{V}$.

Proposition 3. The image of the ellipsoid $E_t(U, Y_0)$ by the affine transformation A is an ellipsoid $\tilde{E}_t(\tilde{U}, \tilde{Y}_0)$ where $\tilde{U} = GUI$.

Proof. $A(E_t(U, Y_0)) = \{\tilde{Y}/\tilde{Y} = GY + a, \|U^{-1}(Y - Y_0)\| = t\} = \{\tilde{Y}/\|(GU)^{-1}(\tilde{Y} - \tilde{Y}_0)\| = t\} = \{\tilde{Y}/\|(\tilde{U}^{-1}(\tilde{Y} - \tilde{Y}_0))\| = t\}$, where $\tilde{U} = GUI$. It follows that $A(E_t(U, Y_0)) = \tilde{E}_t(\tilde{U}, \tilde{Y}_0)$. The semi-axes of the ellipsoid $\tilde{E}_t(\tilde{U}, \tilde{Y}_0)$ will be $\tilde{a}_0(t) = t/\|\tilde{U}^{-1}\| = t/\|(GU)^{-1}\|$, $\tilde{a}_1(t) = t\|\tilde{U}\| = t\|GU\|$ and $\tilde{p}_1 = \|GU\| \|(GU)^{-1}\|$.

Proposition 4. If $E_t(U, Y_0)$ is a sphere then its image by the affine transformation A , $\tilde{E}_t(\tilde{U}, \tilde{Y}_0)$, has the characteristic $\tilde{U} = \|U\| GJI$ and the extreme semi-axes

$$\tilde{a}_0(t) = t \frac{\|U\|}{\|G^{-1}\|}, \quad \tilde{a}_1(t) = t\|G\| \|U\|.$$

Proof. By the hypothesis it follows that $U = \|U\| J$, hence $\tilde{U} = GUI = \|U\| GJI$ and $\tilde{a}_0(t) = t/\|\tilde{U}^{-1}\| = t\|U\| \|G^{-1}\|$, $\tilde{a}_1(t) = t\|\tilde{U}\| = t\|G\| \|U\|$. We observe that $\tilde{p}_1 = \|G\| \|G^{-1}\|$.

Proposition 5. *The affine transformation A maps the ellipsoid $E_t(U, Y_0)$ into a sphere of radius $\tilde{r}$ iff $U = \|GU\|G^{-1}\tilde{J}I^{-1}$ and $t = \tilde{r}/\|GU\|$.*

Proof. $\tilde{E}_t(\tilde{U}, \tilde{Y}_0)$ is a sphere iff $\tilde{U} = \|\tilde{U}\|\tilde{J}$, where $\tilde{J}: \tilde{V} \rightarrow \tilde{V}$ is an isomorphism of vectorial normed spaces. It follows that $\tilde{U} = GUI = \|GUI\|\tilde{J} = \|GU\|\tilde{J}$, hence $U = \|GU\|G^{-1}\tilde{J}I^{-1}$, $\tilde{r} = t\|\tilde{U}\| = t\|GU\|$. The converse is evident.

Remark. The extreme semi-axes of $E_t(U, Y_0)$ are $a_0(t) = t/\|U^{-1}\| = t\|GU\|/\|G\|$, $a_1(t) = t\|U\| = t\|GU\|\|G^{-1}\|$ and $p_1 = \|U\|\|U^{-1}\| = \|G\|\|G^{-1}\|$.

Proposition 6. *The affine transformation A maps spheres into spheres, iff $I_0 = G/\|G\|$ is an isomorphism of vectorial normed spaces.*

Proof. $E(U, Y_0)$ and $\tilde{E}_t(\tilde{U}, \tilde{Y}_0)$ are spheres iff $U = \|U\|\tilde{J}$, $\tilde{U} = \|\tilde{U}\|\tilde{J}$ and $\tilde{U} = GUI$. We have $G\|U\|JI = \|U\|\|G\|\tilde{J}$, hence $G/\|G\| = JI^{-1}J^{-1} = I_0$. The converse is evident.

Remark. If $E_t(U, Y_0)$ is the sphere of radius $r = t\|U\|$, then the sphere $\tilde{E}_t(\tilde{U}, \tilde{Y}_0)$ has the radius $\tilde{r} = r\|G\|$. Indeed $\tilde{r} = t\|\tilde{U}\| = t\|GUI\| = t\|GU\| = t\|G\|\|U\| = \|G\|\|U\| = r\|G\|$.

2. Regular Families of Hipersurfaces and Regular Maps. Let us consider the unit ball $B(0, 1) \subset V$ and the homeomorphism $\varphi: B(0, 1) \rightarrow V$, such that $\varphi(0) = Y_0$. We denote by $\sigma_\alpha = \varphi(S(0, \alpha))$, $\alpha \in (0, 1)$; and we consider the family of hipersurfaces $\sigma = \{\sigma_\alpha / \alpha \in (0, 1)\}$. The limit

$$k_\sigma(Y_0) = \lim_{\alpha \rightarrow 0} \sup \frac{\|Y - Y_0\| / Y \in \sigma_\alpha}{\inf \{\|Y - Y_0\| / Y \in \sigma_\alpha\}}$$

is called the parameter of the family σ .

We say that the family of hipersurfaces σ is regular at Y_0 , if $k_\sigma(Y_0) < \infty$ (see [2]).

If $f: V \rightarrow \tilde{V}$ is a homeomorphism, we denote by $\tilde{\sigma} = \{f(\sigma_\alpha) / \alpha \in (0, 1)\}$. The homeomorphism f is called regular at Y_0 if there exists at least a regular family of hipersurfaces σ at Y_0 such that the family of hipersurfaces $\tilde{\sigma}$ is regular at $\tilde{Y}_0 = f(Y_0)$, hence $\tilde{k}_{\tilde{\sigma}}(\tilde{Y}_0) < \infty$. f is said to be regular in V if it is regular at each of its points. We denote by $\Sigma = \{\sigma / \sigma \text{ is a regular family at } Y_0\}$.

The function $q: V \rightarrow R$ defined by

$$q(Y_0) = \inf \{\tilde{k}_{\tilde{\sigma}}(\tilde{Y}_0) / \sigma \in \Sigma\}$$

is called the characteristic of f .

Remark. If $f: V \rightarrow \tilde{V}$ is a regular homeomorphism such that $q(Y) \leq K < \infty$, then f^{-1} is also a regular homeomorphism and its characteristic $q_1(\tilde{Y}) \leq K$. If, in addition, $g: \tilde{V} \rightarrow \tilde{\tilde{V}}$ is a regular homeomorphism with the characteristic $\tilde{q}(\tilde{Y}) \leq \tilde{K} < \infty$, then $g \circ f$ is a regular homeomorphism and its characteristic $\tilde{\tilde{q}}(Y) \leq K \cdot \tilde{K}$.

Theorem 1. *If $A: V \rightarrow \tilde{V}$ is an affine transformation, $AY = GY + a$, then A is regular and its characteristic $q(Y) = \|G\|\|G^{-1}\|$.*

Proof. Let us consider the family of hipersurfaces $\sigma = \{\sigma_t\} = \{E_t(U, Y_0)\}$, where $U = \|GU\|G^{-1}\tilde{J}I^{-1}$. It follows that $\tilde{k}_{\tilde{\sigma}}(\tilde{Y}_0, k_\sigma(Y_0)) = \|G\|\|G^{-1}\|$. If we consider

the regular family of hypersurfaces $\sigma = \{\sigma_\alpha\}$ with $k_\sigma(Y_0) \geq \|G\| \|G^{-1}\|$, we get $k_\sigma(Y_0) \cdot \tilde{k}_{\tilde{\sigma}}(\tilde{Y}_0) \geq \|G\| \|G^{-1}\|$.

Let $\sigma = \{\sigma_\alpha\}$ be a regular family of hypersurfaces with $k_\sigma(Y_0) < \|G\| \|G^{-1}\|$ and $r_1(\alpha) = \sup\{\|Y - Y_0\| / Y \in \sigma_\alpha\}$, $r_0(\alpha) = \inf\{\|Y - Y_0\| / Y \in \sigma_\alpha\}$. We consider the ellipsoids $E_{t_0}(U, Y_0)$ and $E_{t_1}(U, Y_0)$, $U = \|GU\| G^{-1} \tilde{J}^{-1}$ such that $a_0(t_0) = r_0(\alpha)$ and $a_1(t_1) = r_1(\alpha)$. The image of these ellipsoids are the spheres $\tilde{E}_{t_0}(\tilde{U}, \tilde{Y}_0)$ and $\tilde{E}_{t_1}(\tilde{U}, \tilde{Y}_0)$ of radius $\tilde{r}_0 = t_0 \|GU\|$ and respectively $\tilde{r}_1 = t_1 \|GU\|$.

By a similar argument as in Proposition 1 from [1], we obtain $k_\sigma(Y_0) \cdot \tilde{k}_{\tilde{\sigma}}(\tilde{Y}_0) \geq \|G\| \|G^{-1}\|$. It follows that $q(Y_0) = \inf_{\{\sigma\}} k_\sigma(Y_0) \cdot \tilde{k}_{\tilde{\sigma}}(\tilde{Y}_0) = \|G\| \|G^{-1}\|$.

Let $F: V \rightarrow \tilde{V}$ a homeomorphism with $F(0) = \tilde{0}$, which is Frechet differentiable at 0 and its Frechet differential at 0, $DF(0)$, is an isomorphism of vectorial topological spaces. Let $\sigma = \{\sigma_\alpha\}$ be a regular family of hypersurfaces at 0 and $\tilde{\sigma} = \{F(\sigma_\alpha)\}$, $\tilde{\sigma}_1 = \{DF(0)(\sigma_\alpha)\}$.

Theorem 2. $\tilde{\sigma}$ is regular at $\tilde{0}$, iff $\tilde{\sigma}_1$ is regular at $\tilde{0}$ and then $\tilde{k}_{\tilde{\sigma}}(\tilde{0}) = \tilde{k}_{\tilde{\sigma}_1}(\tilde{0})$.

The proof is similar to the corresponding one given by us in [1], for the Hilbert spaces.

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REFERENCES

1. Borcea V. T., Neagu Al., *Ellipsoids and Regular Mappings in Riemannian Manifolds* (to appear).
2. Caraman P., *Quasiconformal Mappings in Real Normed Spaces*, Rev. Roum. Math. Pures et Appl., **24**, 33–78 (1979).

APLICAȚII REGULATE ÎN SPAȚII FINSLER (I)

(Rezumat)

Scopul acestei Note este de a determina caracteristica unei transformări afine și a unui homeomorfism diferențiabil între două spații vectoriale normate. Aceste rezultate vor fi aplicate în studiul aplicațiilor regulate între spații Finsler.

SOME GEOMETRICAL ASPECTS OF THE THEORY OF QUADRATIC DIFFERENTIAL EQUATIONS

BY

ILIE BURDUJAN

0. In this paper it is shown that for some quadratic differential equations a suitable metric can be associated. The Riemannian structure defined by this metric tensor leads us to consider some suitable coordinates.

1. **Preliminaries.** Let E be a Banach space over the field K (here K is either $\mathbb{R}$ or $\mathbb{C}$), with the norm denoted by $\|\cdot\|_1$. A differential equation in E , having the form

$$(1) \quad \frac{dX}{dz} = F(X),$$

where $F: E \rightarrow E$ is a vectorial quadratic continuous form on E and $X: I \subset K \rightarrow E$ (I being a connected open set of K) is the unknown function, is called a *quadratic differential equation* (briefly QDEq) in E .

If $G: E \times E \rightarrow E$ is the polar form of the quadratic form F , then the Banach space E can be organized as a commutative (nonassociative, in general) algebra with the binary operation defined by

$$(2) \quad x \cdot y = G(x, y), \quad \forall x, y \in E.$$

In this case E can be endowed with a new norm $\|\cdot\|$ satisfying the condition $\|x \cdot y\| \leq \|x\| \cdot \|y\|$, $\forall x, y \in E$.

The normed algebra $E(\cdot)$, endowed with the norm $\|\cdot\|$, will be called a *B-algebra*. Obviously, any associative *B-algebra* is a Banach algebra. When no confusion is possible, this *B-algebra* will be briefly denoted by E .

It is proved in [1] that:

- (i) two QDEqs are equivalent iff the algebras associated to them as above are isomorphic;
- (ii) the solution of the Cauchy problem for a QDEq is an analytical function
- (iii) if the *B-algebra* associated to a QDEq is a power-associative one; the solution of the Cauchy problem, with the initial condition $X(t_0) = x_0$, is,

$$(3) \quad X(t) = [I - (t - t_0)L_{x_0}]^{-1}(x_0),$$

(here $L_{x_0}: x \rightarrow L_{x_0}(x) = x_0 \cdot x$, $\forall x \in E$ is the left multiplication by x_0 and I is the identity operator on E). It was also shown how some qualitative properties of a QDEq are reflected in the properties of its associated *B-algebra*. Mo-

reover, it was established in [1] the practical procedure either for finding the solution of the Cauchy problem for a QDEq in a finite-dimensional Banach space, or for passing to a more simplified form of this problem.

2. In this section it will be proved that any B -algebra used in what follows may be considered to have an identity element. This assumption has the quality to make the subsequent presentation easier. Indeed, if the B -algebra $\mathbf{E}$ has not an identity element, there is a standard construction for obtaining an algebra $\mathbf{E}_1$ which has the identity element 1 , such that $\mathbf{E}_1$ contains an isomorphic copy of $\mathbf{E}$ as an ideal and such that $\mathbf{E}_1/\mathbf{E}$ has dimension 1 over $\mathbf{K}$ (see [3], p. 11).

Let us consider the direct sum $\mathbf{E}_1 = \mathbf{E} \oplus \mathbf{K}$ endowed with the binary operation

$$(x \oplus \lambda) \cdot (y \oplus \mu) = (G(x, y) + \mu x + \lambda y) \oplus (\lambda \mu).$$

$\mathbf{E}_1$ becomes an algebra over $\mathbf{K}$, with the identity element $1 = 0 \oplus 1$ (1 — being the identity element for the multiplication in $\mathbf{K}$). Then, the bilinear form $G' : \mathbf{E}_1 \times \mathbf{E}_1 \rightarrow \mathbf{E}_1$ defined by

$$G'(x \oplus \lambda, y \oplus \mu) = (G(x, y) + \lambda x + \mu y) \oplus (\lambda \mu), \quad \forall x \oplus \lambda, y \oplus \mu \in \mathbf{E}_1$$

may be considered. The vectorial quadratic form $F' : \mathbf{E}_1 \rightarrow \mathbf{E}_1$ associated to G' is

$$F'(x \oplus \lambda) = (F(x) + 2\lambda x) \oplus (\lambda^2), \quad \forall x \oplus \lambda \in \mathbf{E}_1.$$

The quadratic differential system (briefly QDS)

$$\frac{dX}{dt} = F(X) + 2\lambda X, \quad \frac{d\lambda}{dt} = \lambda^2,$$

is associated to the vectorial quadratic form F' . Then, the Cauchy problem

$$(4) \quad \frac{dX}{dt} = F(X) + 2\lambda X, \quad \frac{d\lambda}{dt} = \lambda^2,$$

$$X(t_0) = x_0, \quad \lambda(t_0) = 0,$$

is associated, in a natural way, to the Cauchy problem

$$(5) \quad \frac{dX}{dt} = F(X), \quad X(t_0) = x_0.$$

The connection between the solutions of these two Cauchy problems is: $X(t) \oplus \lambda(t)$ is the solution of the problem (4) iff $X(t)$ is the solution of (5) and $\lambda(t) (\equiv 0)$ is the solution of the Cauchy problem $d\lambda/dt = \lambda^2$, $\lambda(t_0) = 0$. That is why only B -algebras with identity elements will be used in what follows.

Moreover, if $\mathbf{E}(\cdot)$ is an associative (power-associative, alternative, resp.) algebra, then $\mathbf{E}_1(\cdot)$ is also an associative (power-associative, alternative, resp.) algebra.

3. Let us consider the Cauchy problem (5) and assume that its associated B -algebra is an alternative one, with the identity element 1 . Then, any two

elements of such an algebra generate an associative subalgebra, according to Artin's theorem (see [3], p. 29). It should be remarked that the most of the following results are also valid for power-associative algebras. Obviously, any alternative algebra is a power-associative one.

For any power-associative B -algebra E , with the identity element 1 , the powers x^n can be recurrently defined by

$$(6) \quad x^1 = x, \quad x^2 = G(x, x) = F(x), \quad x^n = x^{n-1} \cdot x = G(x^{n-1}, x), \quad \forall n \geq 2, \quad \forall x \in E.$$

Next we introduce the following functions :

$$(7) \quad \text{LN}_B(1-x) = x + \frac{1}{2}x^2 + \dots + \frac{1}{n}x^n + \dots \quad \forall x \in E, \|x\| < 1,$$

$$(8) \quad \text{EXP}_B(x) = 1 + \frac{1}{1!}x + \dots + \frac{1}{n!}x^n + \dots \quad \forall x \in E.$$

Then, the equations

$$(9) \quad \text{LN}_B(\text{EXP}_B(x)) = x, \quad \text{EXP}_B(\text{LN}_B(x)) = x,$$

hold for any $x \in E$, such that the left hand side expressions exist.

Let us consider the Cauchy problem

$$(10) \quad \frac{d^2 X}{dt^2} = F\left(\frac{dX}{dt}\right), \quad X(t_0) = x_0, \quad \frac{dX}{dt}(t_0) = \dot{x}_0.$$

It was recurrently found that

$$(11) \quad X''(t_0) = \dot{x}_0^2; \quad X^{(n)}(t_0) = (n-1)! \dot{x}_0^n \quad (\forall n \in \mathbb{N}^*),$$

(here $X^{(h)}$ denotes $d^h X/dt^h$). The solution of problem (10) has the following Taylor expansion

$$(12) \quad X(t) = x_0 + (t-t_0)\dot{x}_0 + \frac{1}{2}(t-t_0)^2\dot{x}_0^2 + \dots + \frac{1}{n}(t-t_0)^n\dot{x}_0^n + \dots$$

By using (7) for $\|t-t_0\|\|\dot{x}_0\| < 1$, one gets

$$(13) \quad X(t) = x_0 - \text{LN}_B[1 - (t-t_0)\dot{x}_0].$$

Let us now consider the transformation

$$(14) \quad X = x_0 - \text{LN}_B(1-Y),$$

or its equivalent form

$$(14') \quad X = x_0 + Y + \frac{1}{2}Y^2 + \dots + \frac{1}{n}Y^n + \dots$$

Obviously, (14) is locally equivalent to

$$(15) \quad Y = 1 - \text{EXP}_B(x_0 - X).$$

From (14) we obtain

$$\frac{dX}{dt} = (1-Y)^{-1} \frac{dY}{dt}, \quad \frac{d^2X}{dt^2} = (1-Y)^{-1} \frac{d^2Y}{dt^2} + [(1-Y)^{-1}]^2 \left(\frac{dY}{dt} \right)^2,$$

$$F\left(\frac{dX}{dt}\right) = [(1-Y)^{-1}]^2 \left(\frac{dY}{dt} \right)^2.$$

Consequently, the problem (10) becomes

$$(16) \quad \frac{d^2Y}{dt^2} = 0, \quad Y(t_0) = 0, \quad \dot{Y}(t_0) = \dot{x}_0.$$

This problem has the solution

$$(17) \quad Y(t) = (t - t_0) \dot{x}_0$$

i.e., in the new coordinates, the integral curves for (10) are straight lines in $\mathbf{E}$.

The problem (5), in the new coordinates, becomes

$$\frac{d}{dt} [x_0 - \text{LN}_B(1 - Y)] = [x_0 - \text{LN}_B(1 - Y)]^2.$$

Since the B -algebra is a power-associative one, one gets (via (3))

$$(18) \quad Y = 1 - \text{EXP}_{ii} [x_0 - (I - (t - t_0)L_{x_0})^{-1}(x_0)].$$

This form of the solution is suitable for interpretations.

4. The above results are valid for QDEqs in finite-dimensional Banach spaces, too. Besides, in this case, some geometrical interpretations of these results may be given.

Let us consider the case when $\dim \mathbf{E} = n$. Then $\mathbf{E}$ can be identified with $\mathbf{K}^n$. If $\mathcal{B} = \{e_1, \dots, e_n\} \subset \mathbf{K}^{(n)}$ is a basis over $\mathbf{K}$, the Cauchy problem (5) becomes

$$(19) \quad \frac{dx^i}{dt} = a_{jk}^i x^j x^k = 0, \quad x^i(t_0) = x_0^i,$$

where $x = x^i e_i$, $x(t_0) = x_0^i e_i$ and $F(x) = a_{jk}^i x^j x^k e_i$ (i.e. a_{jk}^i are the coefficients of F with respect to the basis $\mathcal{B}$). It is possible to associate with the problem (19) the commutative algebra on $\mathbf{K}^n$ which, in basis $\mathcal{B}$, has the structure constants a_{jk}^i , i.e. $e_j \cdot e_k = a_{jk}^i e_i$.

We also consider the problem

$$(20) \quad \frac{d^2x^i}{dt^2} - a_{jk}^i \frac{dx^j}{dt} \frac{dx^k}{dt} = 0, \quad x^i(t_0) = x_0^i, \quad \frac{dx^i}{dt}(t_0) = \dot{x}_0^i.$$

If the algebra associated to the problem (19) is an alternative one, then, in the same way as for the infinite-dimensional case, it may be built up a local coordinate system such that the problem (20) becomes

$$(21) \quad \frac{d^2y^i}{dt^2} = 0, \quad y^i(t_0) = 0, \quad \dot{y}^i(t_0) = \dot{x}_0^i,$$

which has the solution $y^i = (t - t_0) \dot{x}_0^i$.

In the finite-dimensional case, besides the just presented results, it can be shown that the class of QDS (19), for which the previous result is valid, is larger than that one for which the associated algebra is alternative. In order to prove this assertion, we reformulate the above problem having as starting point the observations that equations (20) look like the equations of the geodesics endowed with an affine connection. The a_{jk}^i 's are multiplication constants of the algebra associated to a QDS, hence they are the components of a (1, 2)-tensor. Therefore, a_{jk}^i 's will be the coefficients of a symmetric connection on the analytic manifold $\mathbf{K}^n$, whose differentiable structure is generated by the maps chosen so that the coordinate transformations in the common parts of two by two domains of maps be linear transformations. Consequently, $\mathbf{K}^n$ endowed with such a structure is a manifold with a constant symmetric connection in the sense of the definition due to G. V r ă n c e a n u [4]. Then, the following (old) problem may be set as follows: Does there exist a Riemannian metric g_{ij} on $\mathbf{K}^n$ such that its Christoffel symbols of the second kind are $-a_{jk}^i$? In its turn, this problem may be reformulate as follows: When asymmetric connection is induced by a Riemannian structure? The solution of this problem is presented in ([2] p. 78). There it is proved that a necessary and sufficient condition that a space with an assigned symmetric connection be Riemannian is that the equations

$$(22) \quad \frac{\partial g_{ij}}{\partial x^k} = -g_{ik}a_{jk}^h - g_{hj}a_{ik}^h,$$

admit a solution. In its turn, a necessary and sufficient condition for that is stated in terms of curvature tensor R_{jkm}^i . The curvature tensor for the space having the symmetric connection with the coefficients $-a_{jk}^i$ is $R_{jkm}^i = a_{jm}^h a_{hk}^i - a_{jk}^h a_{hm}^i$ ($=\text{const.}$). This tensor may be interpreted in terms of the associators of $\mathbf{K}^n$, namely

$$(e_j, e_k, e_m) = e_j e_k \cdot e_m - e_j \cdot e_k e_m = R_{kmj}^i e_i.$$

Consequently, if the algebra is associative, then $R_{jkm}^i = 0$ and the conditions for integrability of (22) are fulfilled. Then, there exists a Riemannian structure on $\mathbf{K}^n$ which is a local euclidean one. A coordinate system can be also chosen for which the coefficients of the connection vanish in a neighborhood of a chosen point $P(x_0^i)$. In such a coordinate system the solutions g_{ij} of (22) are constants.

In a similar way as in the infinite-dimensional case, if the associated algebra is alternative, then there exists a coordinate system such that the „geodesics“ has the equations (21), that is, a coordinate system for which the coefficient of the connection vanish. In this case, the system (22) becomes

$$(23) \quad \frac{\partial g_{ij}}{\partial y^k} = 0,$$

and consequently it is completely integrable.

Then, the following question is arising: How large is the class of differential system (19) such that we have assured the existence of a metric g_{ij} whose Christoffel symbols of second kind are $-a_{jk}^i$? Certainly, this class contains the class of algebras associated to Riemannian spaces studies by V.

Vagner [5]. It should be remarked that there exists such spaces for which the associated algebra is not even power-associative algebras.

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REFERENCES

1. Burdujan I., *Observații asupra sistemelor de ecuații diferențiale pătratice cu coeficienți constanți* (I), (II). *Lucrările Sesiunii Științifice Jubiliare, Inst. Polit. Iași*, pp. 58–61, 62–65, 1988.
2. Eisenhart L. P., *Non-Riemannian Geometry*. Amer. Math. Soc. Coll. Publ., VIII, New York, 1927.
3. Schaffer R. D., *An Introduction to Nonassociative Algebras*. Acad. Press, New York–London, 1966.
4. Vranceanu G., *Lecții de geometrie diferențială*. Vol. II, Ed. did. și ped., Buc., 1964.
5. Вaгнер В., *Римановы многообразия с постоянными символами Кристоффеля*. *Бюл. Саратовский университет*, 15–32 (1938).

ASPECTE GEOMETRICE ALE TEORIEI ECUAȚIILOR DIFERENȚIALE PĂTRATICE

(Rezumat)

Se arată că pentru anumite ecuații diferențiale pătratice se poate asocia o metrică specifică. Structura riemanniană definită de acest tensor metric conduce la considerarea unor sisteme de coordonate convenabile.

SOME REMARKS ON A QUASILINEAR DIFFERENTIAL SYSTEM

BY

ADRIAN CORDUNEANU and GH. ZAHARIA

Consider the differential system

$$(1) \quad y'(t) = Ay + g(t, y) + f(t), \quad t \geq 0,$$

where

(H₁) A is a constant real matrix of type $n \times n$, whose eigenvalues λ_k have negative real part: ($\operatorname{Re} \lambda_k < 0$ for $k = \overline{1, n}$);

(H₂) $g: \mathbb{R}_+ \times \mathbb{R}_n \rightarrow \mathbb{R}_n$ is a continuous and local-lipschitzian with respect to y vector-function, and there exists $\varphi \in L^1(\mathbb{R}_+)$ such that

$$(2) \quad \|g(t, y)\| \leq \varphi(t) \|y\| \quad \text{for } t \geq 0, y \in \mathbb{R}_n; \quad \int_0^\infty \varphi(t) dt < \infty,$$

where $\varphi = \varphi(t) \geq 0$ is a continuous scalar function;

(H₃) $f: \mathbb{R}_+ \rightarrow \mathbb{R}_n$ is a continuous vector-function, satisfying the condition

$$(3) \quad \lim_{t \rightarrow \infty} f(t) = l \in \mathbb{R}_n.$$

In this paper, we denote by $\mathbb{R}_n$ the space of all constant real vectors of type $n \times 1$, endowed with the norm $\|y\| = \sum_{k=1}^n |y_k|$ for every $y \in \mathbb{R}_n$. The corresponding matrix norm $\|A\|$ will be given then by the formula

$$(4) \quad \|A\| = \max_j \left\{ \sum_{i=1}^n |a_{ij}| \right\}.$$

We can state the following

Proposition 1. *Under the cited above hypotheses (H₁), (H₂) and (H₃), every noncontinuable solution $y = y(t)$ of (1) is defined for $t \geq 0$ and there exists*

$$(5) \quad \lim_{t \rightarrow \infty} y(t) = -A^{-1}l.$$

Moreover, if $\varphi(t) \rightarrow 0$ for $t \rightarrow \infty$, we also have $y'(t) \rightarrow 0$ for $t \rightarrow \infty$.

Proof. From (1), using the variation of constants formula, we obtain

$$(6) \quad y(t) = e^{At}c + \int_0^t e^{A(t-s)} [g(s, y(s)) + f(s)] ds,$$

where $c \in \mathbb{R}_n$ is an arbitrary constant vector. Writting $f(t) = l + u(t)$ with $u(t) \rightarrow 0$ as $t \rightarrow \infty$, we have

$$(7) \quad y(t) = e^{At}(c + A^{-1}l) - A^{-1}l + \int_0^t e^{A(t-s)}g(s, y(s))ds + \int_0^t e^{A(t-s)}u(s)ds,$$

which permits us to show that each noncontinuable solution $y = y(t)$ remains bounded and consequently is defined (may be prolonged) on the whole semi-axis $t \geq 0$. Indeed, being well-known that, under the hypothesis (H_1) , there exist two constants $K > 0$ and $\alpha > 0$, such that

$$(8) \quad \|e^{At}\| \leq Ke^{-\alpha t}, \quad t \geq 0,$$

we obtain from (7) the inequality

$$(9) \quad \|y(t)\| \leq K_1 + K \int_0^t e^{-\alpha(t-s)}\varphi(s)\|y(s)\|ds + K \int_0^t e^{-\alpha(t-s)}\|u(s)\|ds,$$

where $K_1 > 0$ is a constant. Because $\lim_{t \rightarrow \infty} u(t) = 0$, we get

$$(10) \quad \int_0^t e^{-\alpha(t-s)}\|u(s)\|ds \rightarrow 0, \quad \text{when } t \rightarrow \infty,$$

and therefore

$$(11) \quad \|y(t)\| \leq K_2 + K \int_0^t e^{-\alpha(t-s)}\varphi(s)\|y(s)\|ds,$$

where $K_2 > 0$ is a constant, depending only on $c \in \mathbb{R}_n$ appearing in (6). This inequality being valid on the maximum interval $[0, t_0)$, with $t_0 > 0$, then we have

$$(12) \quad \|y(t)\| \leq K_2 + K \int_0^t \varphi(s)\|y(s)\|ds$$

on the same interval and using Gronwall-Bellman lemma, we obtain

$$(13) \quad \|y(t)\| \leq K_2 \exp \left(K \int_0^t \varphi(s)ds \right),$$

which implies that every solution $y = y(t)$ of (1) is defined and bounded for $t \geq 0$. Of course, the constant K_2 depends of the given solution $y = y(t)$. From the majoration

$$(14) \quad \left\| \int_0^t e^{A(t-s)}g(s, y(s))ds \right\| \leq K_3 \int_0^t e^{-\alpha(t-s)}\varphi(s)ds, \quad t \geq 0,$$

where $K_3 > 0$ is a constant. Since $\varphi \in L^1(\mathbb{R}_+)$ and $\lim_{t \rightarrow \infty} e^{-at} \varphi(t) = 0$, therefore

$$(15) \quad \int_0^t e^{A(t-s)} g(s, y(s)) ds \rightarrow 0 \quad \text{when } t \rightarrow \infty.$$

Then, making $t \rightarrow \infty$ in (7), we obtain the relation (5). If $\varphi(t) \rightarrow 0$ when $t \rightarrow \infty$ we obtain $y'(t) \rightarrow 0$ when $t \rightarrow \infty$, letting $t \rightarrow \infty$ in the Eq. (1). Compare with Theorem 3.1, Ch. XIII of [1].

Remark 1. Under the hypotheses (H_1) , (H_3) and the condition

$$(16) \quad \int_0^\infty \|B(t)\| dt < \infty,$$

where $B = B(t)$ is a continuous real $n \times n$ matrix, the conclusion (5) of the preceding Proposition 1 holds for the system

$$(17) \quad y'(t) = (A + B(t))y + f(t), \quad t \geq 0.$$

Compare with the problem 32 in [2], p. 233.

Consider the quasilinear n -th order differential equation

$$(18) \quad y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = g(t, y, y', \dots, y^{(n-1)}) + f(t), \quad t \geq 0,$$

where $a_i, i = \overline{1, n}$ are real constants and g, f are scalar real functions and assume that

(H_4) the roots of the polynomial $P(\lambda) = \lambda^n + a_1 \lambda^{n-1} + \dots + a_n$ have negative real part;

(H_5) $g: \mathbb{R}_+ \times \mathbb{R}_n \rightarrow \mathbb{R}$ (continuous and local-lipschitzian) function satisfying the condition

$$(19) \quad |g(t, y_1, y_2, \dots, y_n)| \leq \varphi(t) \sum_{k=1}^n |y_k|, \quad \int_0^\infty \varphi(t) dt < \infty,$$

for $t \geq 0, y_i \in \mathbb{R}, i = \overline{1, n}$ where $\varphi = \varphi(t) \geq 0$ is a continuous scalar function;

(H_6) $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ is a continuous scalar function, having the property

$$(20) \quad \lim_{t \rightarrow \infty} f(t) = l = \text{finite}.$$

Proposition 2. Under the hypotheses (H_4) , (H_5) , (H_6) , every non-continuable solution $y = y(t)$ of (18) is defined for $t \geq 0$ and there exist

$$(21) \quad \lim_{t \rightarrow \infty} y(t) = \frac{l}{a_n}, \quad \lim_{t \rightarrow \infty} y^{(k)}(t) = 0 \quad \text{for } k = \overline{1, n-1}.$$

Moreover, if $\varphi(t) \rightarrow 0$ as $t \rightarrow \infty$, we also have $y^{(n)}(t) \rightarrow 0$ as $t \rightarrow \infty$.

Proof. Putting $y_1 = y, y_2 = y', \dots, y_n = y^{(n-1)}$ and $Y^T = [y_1, y_2, \dots, y_n]$, Eq. (18) is equivalent with the system

$$(22) \quad Y'(t) = AY + G(t, Y) + F(t), \quad t \geq 0,$$

where

$$(23) \quad A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}, \quad G = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ g \end{bmatrix}, \quad F = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f \end{bmatrix}.$$

We are in the case of the preceding Proposition 1 and hence there exists

$$(24) \quad \lim_{t \rightarrow \infty} Y(t) = Y_\infty = -A^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ l \end{bmatrix} \Leftrightarrow AY_\infty = - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ l \end{bmatrix}.$$

The last matricial equality implies that the relations (21) are true for every solution $y=y(t)$ of Eq. (18). If suppose $\varphi(t) \rightarrow 0$ as $t \rightarrow \infty$, we have $Y'(t) \rightarrow 0$ as $t \rightarrow \infty$ and then $y^{(n)} \rightarrow 0$ as $t \rightarrow \infty$, also for every solution of (18).

Remark 2. Under the hypotheses (H_4) , (H_6) and the condition

$$(25) \quad \int_0^\infty |g_k(t)| dt < \infty, \quad k = \overline{1, n},$$

where $g_k(t)$ are continuous real functions, the conclusion (21) of the Proposition 2 holds for the equation

$$(26) \quad y^{(n)} + (a_1 + g_1(t))y^{(n-1)} + \dots + (a_n + g_n(t))y = f(t), \quad t \geq 0.$$

Moreover, if we suppose $g_n(t) \rightarrow 0$ as $t \rightarrow \infty$ and $g_k(t) = \text{bounded}$, $k = \overline{1, n-1}$ it also follows $y^{(n)}(t) \rightarrow 0$ as $t \rightarrow \infty$ for every noncontinuable solution of (26).

Let us again consider the system (1) under the hypotheses (H_1) and (H_2) and assume that the vector-function $f=f(t)$ satisfies the conditions (H_7) $f: \mathbb{R}_+ \rightarrow \mathbb{R}_n$ has, on any compact interval, at most a finite number of discontinuities with finite or infinite lateral limits and there exists $M > 0$, such that

$$(27) \quad \int_t^{t+1} \|f(s)\| ds \leq M, \quad t \geq 0;$$

(H_8) there exists „the mean of“ the vector-function f , i. e. exists

$$(28) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(s) ds = \bar{f} \in \mathbb{R}_n.$$

Proposition 3. Assume that the hypotheses (H_1) , (H_2) , (H_7) and (H_8) are fulfilled. Then, every noncontinuable solution $y=y(t)$ of (1) is defined and bounded for $t \geq 0$ and there exists

$$(29) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t y(s) ds = -A^{-1} \bar{f} \in \mathbb{R}_n.$$

P r o o f. For a fixed solution $y=y(t)$ of (1), making use of the formula (6), we obtain

$$(30) \quad \|y(t)\| \leq \|e^{At}\| \|c\| + K \int_0^t e^{-\alpha(t-s)} \varphi(s) \|y(s)\| ds + K \int_0^t e^{-\alpha(t-s)} \|f(s)\| ds$$

on some positive interval $[0, t_0)$. On the other hand, it is well-known that the condition (27) implies

$$(31) \quad \int_0^t e^{-\alpha(t-s)} \|f(s)\| ds \leq M_1 \quad \text{for } t \geq 0,$$

where $M_1 > 0$ is a constant. Thus, we can use the same argument as in the proof of Proposition 1, to obtain an inequality of the form

$$(32) \quad \|y(t)\| \leq K_4 \exp \left(K \int_0^\infty \varphi(s) ds \right), \quad t \geq 0,$$

where the positive constant K_4 depends of the given solution $y=y(t)$. To prove (29), we start from (6) and we firstly remark that

$$(33) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T e^{At} c dt = 0 \in \mathbb{R}_n.$$

With $T > 0$ fixed, we have

$$(34) \quad \begin{aligned} \int_0^T \left(\int_0^t e^{A(t-s)} g(s, y(s)) ds \right) dt &= \int_0^T \left(\int_s^T e^{A(t-s)} dt \right) g(s, y(s)) ds = \\ &= A^{-1} \int_0^T (e^{A(T-s)} - I) g(s, y(s)) ds, \end{aligned}$$

where I is the unity matrix of type $n \times n$ and consequently

$$(35) \quad \begin{aligned} &\frac{1}{T} \left\| \int_0^T \left(\int_0^t e^{A(t-s)} g(s, y(s)) ds \right) dt \right\| \leq \\ &\leq \frac{\|A^{-1}\|}{T} \left\{ K \int_0^T e^{-\alpha(T-s)} \varphi(s) \|y(s)\| ds + \int_0^T \varphi(s) \|y(s)\| ds \right\}. \end{aligned}$$

Taking into account that $\|y(t)\|$ is bounded for $t \geq 0$, it is easy to see that the left-hand side in (35) tends to zero as $T \rightarrow \infty$. Finally, we have

$$(36) \quad \int_0^T \left(\int_0^t e^{A(t-s)} f(s) ds \right) dt = A^{-1} \int_0^T e^{A(T-s)} f(s) ds - A^{-1} \int_0^T f(s) ds,$$

from which we get

$$(37) \quad \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \left(\int_0^t e^{A(t-s)} f(s) ds \right) dt = -A^{-1} \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(s) ds = -A^{-1} \bar{f}.$$

Concluding, we can state that (29) was proved and moreover

$$(38) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t y'(s) ds = \lim_{t \rightarrow \infty} \frac{y(t) - y(0)}{t} = 0 \in \mathbb{R}_n$$

for every solution $y = y(t)$ of (1).

Remark 3. Under the hypotheses (H_1) , (H_7) , (H_8) and the condition (16), the conclusion of the Proposition 3 holds for the system

$$(39) \quad y'(t) = (A + B(t))y + f(t), \quad t \geq 0.$$

Consider again Eq. (18) and assume that the scalar function f satisfies the conditions analogous to (H_7) and (H_8) , namely

(H_9) $f: \mathbb{R}_+ \rightarrow \mathbb{R}$ has, on any compact interval, at most a finite number of discontinuities with finite or infinite lateral limits and there exists $M > 0$, such that

$$(40) \quad \int_t^{t+1} |f(s)| ds \leq M, \quad t \geq 0;$$

(H_{10}) there exists

$$(41) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t f(s) ds = \bar{f} = \text{finite}.$$

Proposition 4. Assume that (H_4) , (H_5) , (H_9) and (H_{10}) are satisfied. Then, every noncontinuable solution $y = y(t)$ of the equation (18) is defined and bounded for $t \geq 0$, possesses bounded continuous derivatives of the order $\leq n-1$ and

$$(42) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t y(s) ds = \frac{\bar{f}}{a_n}, \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t y^{(k)}(s) ds = 0, \quad k = \overline{1, n}.$$

The proof is similar to that of the Proposition 2. Using the same notations, we observe that for the system (22), the hypotheses of the preceding Proposition 3 are fulfilled and hence, we have

$$(43) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t Y(s) ds = \bar{Y} = -A^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ f \end{bmatrix} \Leftrightarrow A \bar{Y} = - \begin{bmatrix} 0 \\ \vdots \\ 0 \\ \bar{f} \end{bmatrix},$$

where A is the matrix defined by the relation (23). The last matriceal equality in (43) implies that

$$(44) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t y(s) ds = \bar{f}, \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t y^{(k)}(s) ds = 0, \quad k = \overline{1, n-1};$$

but, according to the relation (38), we obtain that the last equality is also true and for $k=n$. This concludes the proof. We point out that the discontinuities produced by the free term $f=f(t)$ occur only for the derivative $y^{(n)}(t)$, which has finite or infinite lateral limits in the points where f is not continuous.

Remark 4. Under the hypotheses (H_4) , (H_9) , (H_{10}) and the conditions (25), the conclusion (42) of the Proposition 4 holds true for the linear equation (26).

Remark 5. The hypotheses (H_9) and (H_{10}) are independent each of other. For example, the function $f(t)=\sqrt{t} \cos t$ satisfies (H_{10}) and does not satisfy (H_9) . On the other hand, we can easily construct a function $f=f(t)$, defined for $t \geq 0$, which satisfies (H_9) and does not satisfy (H_{10}) , putting successively $f(t)=1$ or $f(t)=-1$ on some suitable intervals; we omit the details.

We point out that in the case $g=0$, we can renounce to the conditions (27) or respectively (40), in the Propositions 3 and 4. The assumption that $f=f(t)$ may have the described type of discontinuities is maintained, but we suppose that

(H_{11}) $f=f(t)$ is absolutely integrable on any compact interval of the semi-axis $t \geq 0$.

Proposition 5. Assume that (H_1) , (H_8) and (H_{11}) are fulfilled. Then, every noncontinuable solution $y=y(t)$ of the system

$$(45) \quad y'(t) = Ay + f(t), \quad t \geq 0$$

is defined for $t \geq 0$ and the relation (29) remains true.

Proof. For a fixed solution $y=y(t)$ of (45), we have

$$(46) \quad y(t) = e^{At}c + \int_0^t e^{A(t-s)}f(s)ds,$$

where c is a constant vector of type $n \times 1$. We deduce that $y=y(t)$ remains bounded on any compact interval and therefore is defined for all $t \geq 0$. Taking into account (33), our aim is to show that (37) is also true under the new hypotheses. If we remember (36), it remains to prove only the following relation

$$(47) \quad \lim_{t \rightarrow \infty} \frac{1}{t} \int_0^t e^{A(t-s)}f(s)ds = 0 \in \mathbb{R}_n.$$

To this end, we introduce the continuous vector-function

$$(48) \quad h(t) = \frac{1}{t} \int_0^t f(s) ds, \quad t > 0$$

and we remark that $h(t) \rightarrow \bar{f} \in \mathbb{R}_n$ when $t \rightarrow \infty$, consequently we have $h(t) = \bar{f} + u(t)$, where $u(t) \rightarrow 0$ for $t \rightarrow \infty$. We also have $f(t) = (th(t))'$ in any point t where f is continuous and hence we can write, integrating by parts

$$(49) \quad \begin{aligned} \int_0^t e^{A(t-s)} f(s) ds &= \int_0^t e^{A(t-s)} (sh(s))' ds = th(t) + \\ &+ A \int_0^t e^{A(t-s)} s \bar{f} ds + A \int_0^t e^{A(t-s)} su(s) ds, \quad t > 0. \end{aligned}$$

From (49), we obtain

$$(50) \quad \frac{1}{t} \int_0^t e^{A(t-s)} f(s) ds = h(t) - \bar{f} + \frac{A^{-1}}{t} (e^{At} - I) \bar{f} + \frac{A}{t} \int_0^t e^{A(t-s)} su(s) ds,$$

which easily implies (47). This concludes the proof.

Adapted to the case of the n -th order differential equation

$$(51) \quad y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y = f(t), \quad t \geq 0$$

with constant coefficients, the preceding Proposition 5 takes the following form,

Proposition 6. Assume that (H_4) is fulfilled and the scalar function $f=f(t)$ satisfies (H_{10}) and (H_{11}) . Then, every noncontinuable solution $y=y(t)$ of (51) is defined for $t \geq 0$, possesses continuous derivatives of the order $\leq n-1$ and the relations (42) are true.

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REFERENCES

1. Coddington E. A., Levinson N., *Theory of Ordinary Differential Equations*. McGraw-Hill Book Comp., Inc., New York-Toronto-London, 1955.
2. Демидович Б. П., *Лекции по математической теории устойчивости*. Изд. „Наука“, Москва, 1967.

CÎTEVA OBSERVAȚII ASUPRA UNUI SISTEM DIFERENȚIAL CVASILINIAR

(Rezumat)

Se studiază comportarea soluției pentru $t \rightarrow \infty$ a sistemului $y' = Ay + g(t, y) + f(t)$, $t \geq 0$, în care A este o matrice constantă ale cărei autovalori au partea reală negativă, vector funcția $g=g(t, y)$ satisface majorarea $\|g(t, y)\| \leq \varphi(t)\|y\|$ cu $\int_0^\infty \varphi(t) dt < \infty$ și se urmărește modul în care proprietățile termenului liber $f=f(t)$ se transmit soluției; este vorba de cazul cînd f are limită finită pentru $t \rightarrow \infty$ sau cînd f are „medie“. Se fac aplicații la ecuația cvasiliniară de ordinul n .

THE FINITE ELEMENT METHOD FOR WEAKLY ELLIPTICAL LINEAR EQUATIONS

BY

ARIADNA-LUCIA PLETEA

0. Introduction. The paper deals with the applying of the finite element method to weakly elliptical linear equation considered on bounded domain in $\mathbb{R}^2$. This class of equations comprises elliptical equations (of even order) and a certain class of equations with partial differentials of odd order. The class of weakly elliptical linear operators which generates equations of this type is part of the K -positively defined linear operators introduced by Martynyuk and Petrishyn [3], [5] and studied by G. Dină [2]. Theorems 4.1—4.3 from [2] provide, under circumstances, the existence and uniqueness of the solution for weakly elliptical equations.

1. Variational Formulations for Weakly Elliptical Equations of Even Order. Let be $\Omega \subset \mathbb{R}^2$ a bounded and open set with a continuous Lipschitz bound. The problem is

$$(1.1) \quad Au = f,$$

where

$$(1.2) \quad Au = \sum_{i,k=0}^s (-1)^s \frac{\partial^s}{\partial x^{k-i} \partial y^i} \left(a_{ik}(x, y) \frac{\partial^s u}{\partial x^{s-k} \partial y^k} \right) + a_0(x, y)u,$$

with

$$(1.3) \quad \Omega_A = \left\{ u \in C^{2s}(\Omega); \frac{\partial^k u}{\partial x^{k-i} \partial y^i} \Big|_{\partial\Omega} = 0, k=0, s-1, i=\overline{0, k} \right\}.$$

We impose the following conditions to the coefficients of A :

$$(1.4) \quad \begin{aligned} a_{ik} &\in C^{2s}(\Omega), \quad a_{ik}(x, y) = a_{ki}(x, y), \quad (x, y) \in \Omega, \quad k, i = \overline{0, s}, \\ a_0 &\in C^0(\Omega) \text{ and } a_0(x, y) > 0, \quad (x, y) \in \Omega; \end{aligned}$$

there is μ^2 so that

$$(1.5) \quad \sum_{i,k=0}^s a_{ik}(x, y) \zeta_i \zeta_k \geq \mu^2 \left(\sum_{i,k=0}^s C_s^k \zeta_k \right)^2, \quad (x, y) \in \Omega,$$

$$\zeta = (\zeta_0, \zeta_1, \dots, \zeta_p) \in \mathbb{R}^{p+1}.$$

We suppose that $f \in L^2(\Omega)$. We note

$$H = W^{s,2}(\Omega) = H^s(\Omega),$$

$$H_0 = \left\{ v \in H^s(\Omega) ; \frac{\partial^k u}{\partial x^{k-1} \partial y^i} \Big|_{\partial\Omega} = 0, k=0, s-1, i=\overline{0, k} \right\}.$$

We define the bilinear form $a : H \times H \rightarrow \mathbb{R}$,

$$(1.6) \quad a(u, v) = \int_{\Omega} \left(\sum_{i,k=0}^s a_{ik}(x, y) \frac{\partial^s u}{\partial x^{s-i} \partial y^i} \frac{\partial^s v}{\partial x^{s-k} \partial y^k} + a_0(x, y) uv \right) dx dy$$

and the linear form $f : H \rightarrow \mathbb{R}$,

$$(1.7) \quad (f, v) = \int_{\Omega} f v dx dy.$$

We shall look for the generalized solution of the problem (1.1)–(1.5) according to Definition 4 [1]. This also a solution to the variational problem

$$(1.8) \quad \text{let us find a } u \in H_0 \text{ so that } a(u, v) = (f, v), v \in H_0.$$

The following theorem establishes the relation between the generalized solution and the classical one.

Theorem 1. *The variational problem (1.8) has a unique solution and it is equivalent to the problem (1.1)–(1.5).*

Proof. For proving the existence and uniqueness of the solution to the problem (1.8) we shall use Lax-Milgram's lemma [1]. We show that hypotheses of this lemma are satisfied.

Evidently H_0 is a linear subspace of $H^s(\Omega)$ and as Ω is bounded, the seminorm $|\cdot|_s$ is a norm on the space H_0 , equivalent to the norm $\|\cdot\|_s$ [1]. So H_0 is a Hilbert space.

The bounding of the linear form (1.6) results by using Hölder and Friedrichs–Poincaré's inequalities and the H_0 -ellipticity results from the weak ellipticity of the operator (1.2). So the problem (1.8) has a unique solution.

The equivalence of the problems (1.8) and (1.1)–(1.5) is obtained by using Green's formula several times, the conditions which satisfy the functions of Ω_A and the fact that H_0 is dense in H .

Observation. Condition (1.5) also includes the case of differential operators, whose coefficients satisfy an elliptical condition, that is elliptical operators.

2. Variational Formulations for Weakly Elliptical Equations of Odd Order. Let be $\Omega \subset \mathbb{R}^2$ a bounded and open set with a continuous Lipschitz-bound. We consider the problem

$$(2.1) \quad Bu = f,$$

where

$$(2.2) \quad Bu = \sum_{i=0}^{s-1} \sum_{k=0}^s (-1)^{s-1} \frac{\partial^{s-1}}{\partial x^{s-1-i} \partial y^i} \left(a_{ik}(x, y) \frac{\partial^s u}{\partial x^{s-k} \partial y^k} \right) + a_1(x, y) \delta u,$$

with

$$(2.3) \quad \Omega_B = \left\{ u \in C^{2s-1}(\Omega), \frac{\partial^k u}{\partial x^{k-i} \partial y^i} = 0, k = \overline{1, s-1}, i = \overline{0, k} \right\}.$$

We impose the following conditions on the coefficients of B :

$$(2.4) \quad a_{ik} \in C^{s-1}(\Omega), i = \overline{0, s-1}, k = \overline{0, s}, a_1 \in C^0(\Omega), a_1(x, y) > 0, (x, y) \in \Omega.$$

We note

$$b_{ik} = \begin{cases} a_{0k} & \text{if } i=0 \\ a_{s-1, k} & \text{if } i=s \\ a_{i-1, k} + a_{ik} & \text{if } 1 \neq i \neq s. \end{cases}$$

We suppose that $b_{ik}(x, y) = b_{ki}(x, y)$, $i, k = \overline{0, s}$, $i \leq k$ and

$$(2.5) \quad \sum_{i, k=0}^s b_{ik}(x, y) \zeta_i \zeta_k \geq \mu^2 \left(\sum_{k=0}^s C_s^k \zeta_k \right)^2$$

and, in addition, $f \in L^2(\Omega)$. We note $H = W^{s,2}(\Omega) = H^s(\Omega)$,

$$H_0 = \left\{ v \in H^s(\Omega), \frac{\partial^k v}{\partial x^{k-i} \partial y^i} = 0, k = \overline{1, s-1}, i = \overline{0, k} \right\}.$$

We define the linear form $f: H \rightarrow \mathbb{R}$,

$$(2.6) \quad (f, \delta v) = \int_{\Omega} f \delta v dx dy$$

and the bilinear form $a: H \times H \rightarrow \mathbb{R}$

$$(2.7) \quad a(u, v) = \int_{\Omega} \left(\sum_{i, k=0}^s b_{ik}(x, y) \frac{\partial^s u}{\partial x^{s-i} \partial y^i} \frac{\partial^s v}{\partial x^{s-k} \partial y^k} + a_1(x, y) \delta u \delta v \right) dx dy.$$

We shall look for the generalized solution, according to the same definition out of [1] for the problem (2.1)–(2.5). This is also the solution of the variational problem

$$(2.8) \quad \text{let us find } u \in H_0 \text{ so that } a(u, v) = (f, \delta v), v \in H_0.$$

The following theorem establishes the relation between the generalized solution and the classical one.

Theorem 2. *The variational problem and (2.8) has a unique solution A and it is equivalent to the problem (2.1)–(2.5).*

The proof follows the stages of the proof to Theorem 1. For proving the bordering of the bilinear form (2.7) we use both the quoted inequalities and the fact that

$$\|\delta u\|_{L^2(\Omega)} \leq C \|u\|_{H^1(\Omega)}.$$

3. The Approximation of Weakly Elliptical Variational Problems. An Abstract Evaluation of Error. We consider problems (1.8) and (2.8). We consider H_h , $h \in \mathcal{H}$, $\mathcal{H} \subset \mathbb{R}$, a space dimensionally finite, which ensures an internal approximation of H_0 (the conform finite element method).

We associate to problems (1.8) respectively (2.8) the discrete problems

$$(3.1) \quad \text{find } u_h \in H_h \text{ so that } a(u_h, v_h) = (f, v_h), \quad v_h \in H_h,$$

and respectively

$$(3.2) \quad \text{let us find } u_h \in H_h \text{ so that } a(u_h, v_h) = (f, \delta v_h), \quad v_h \in H_h.$$

We shall say that the family of discrete problems is convergent if, for any variational problem set in the space H_0 , we have $\lim_{h \rightarrow 0} \|u - u_h\| = 0$, where $\|\cdot\|$ means the norm in H_0 .

The following theorem evaluates the error $\|u - u_h\|$ between the solutions of the problems (1.8), (2.8) and respectively (3.1), (3.2).

Theorem 3. (Céa's lemma). *There is a constant C , independent of the subspace H_h so that*

$$(3.3) \quad \|u - u_h\| \leq C \inf_{v_h \in H_h} \|u - v_h\|.$$

So, a sufficient condition of convergence is the existence of a family (H_h) of subspaces of H_0 so that, for any $u \in H$,

$$(3.4) \quad \lim_{h \rightarrow 0} \inf_{v_h \in H_h} \|u - v_h\| = 0.$$

The proof of the theorem for problem (1.8) with the discrete variational problem (3.1) is the same as in [1], [4]. That is why we prove the theorem only for problem (2.8) with the associated discrete problem (3.2).

Let be $w_h \in H_h \subset H_0$ a random element. Out of (1.8) and (3.2) we get

$$a(u, w_h) = (f, \delta w_h), \quad w_h \in H_h \subset H_0, \quad a(u_h, w_h) = (f, \delta w_h), \quad w_h \in H_h.$$

We subtract these equalities and we get

$$(3.5) \quad a(u - u_h, w_h) = 0, \quad w_h \in H_h.$$

First we prove that

$$(3.6) \quad a(u - u_h, u - u_h) = a(u - u_h, u - v_h), \quad w_h \in H_h.$$

This results by applying (3.5) in

$$\begin{aligned} a(u - u_h, u - u_h) &= a(u - u_h, u) - a(u - u_h, u) = \\ &= a(u - u_h, u) - a(u - u_h, v_h) = a(u - u_h, u - v_h). \end{aligned}$$

Using (3.6) we get

$$\alpha_1^2 \|u - u_h\|^2 \leq a(u - u_h, u - u_h) = a(u - u_h, u - v_h) \leq M \|u - u_h\| \|u - v_h\|,$$

where we took into consideration the coercivity of the bilinear form and its bounding. Thus,

$$\|u - u_h\| \leq \frac{M}{\alpha_1^2} \|u - v_h\|$$

and we get the conclusion of the theorem with $C = M/\alpha_1^2$.

4. Error Evaluation for Weakly Elliptical Problems of Order $2s$, Respectively $2s-1$, on Polygonal Domains (the Bidimensional Case). We deal with the convergence of the solution u of the problem (3.1) respectively (3.2) introduced in the previous paragraph with the solution u of the corresponding problem (1.8), (2.8), in case its order is $2s$, respectively $2s-1$.

Let be Ω a polygonal domain in $\mathbb{R}^2$ and $\mathcal{T}_h$ an admissible triangulation of Ω so that $\bar{\Omega} = \bigcup_{T \in \mathcal{T}_h} T$. We associate the following dimensionally finite space to a triangulation $\mathcal{T}_h$

$$H_h = \left\{ u_h; u_h \in C^s(\Omega); \frac{\partial^k u_h}{\partial x^{k-i} \partial y^i} \Big|_{\partial\Omega} = 0, k = \overline{0, s-1}, i = \overline{0, k}, \right. \\ \left. T \in \mathcal{T}_h, u_h \Big|_T \in P_k(T) \right\}.$$

If the problem is of order $2s$, and

$$H_h = \left\{ u_h; u_h \in C^s(\Omega); \frac{\partial^k u_h}{\partial x^{k-i} \partial y^i} = 0, k = \overline{1, s-1}, i = \overline{0, k}, \right. \\ \left. u \Big|_{\Gamma_1} = g, T_1 \subset \partial\Omega, u_h \Big|_T \in P_k(T) \right\}.$$

Theorem 4. Let be (T, Σ, P) , $T \in \mathcal{T}_h$, $h \in \mathcal{H}$ a regular family of finite element of class C^{s-1} with its reference element $(\hat{T}, \hat{\Sigma}, \hat{P})$ and σ the highest differentiation order which appears in defining the set Σ . We suppose there is a $k \geq s$ so that

$$(4.1) \quad P(\hat{T}) \subset \hat{P} \subset H^s(\hat{T}),$$

$$(4.2) \quad H^{k+1}(\hat{T}) \hookrightarrow C^\sigma(\hat{T}).$$

If the solution u of the problem (1.1) or (2.1) is in $H^k(\Omega)$ and u_h is the solution to the corresponding problem (3.1) respectively (3.2), then there is a constant C independent of h so that

$$(4.3) \quad \|u - u_h\|_{s, \Omega} \leq Ch^{k+1-s} |u|_{k+1, \Omega}, \quad h \in \mathcal{H}.$$

Proof. As the family of finite element (T, Σ, P) , $T \in \mathcal{T}_h$ is of class C^{s-1} , it results that $H_h \subset C^{s-1}(\Omega)$, and from $P^k \subset H^s(\hat{T})$ it results $H_h \subset H^s(\Omega)$. According to Theorem 2.10.3 [4] with $p=q=2$ and for $m=0, 1, \dots, s$

$$|u - \pi_T u|_{0, T} \leq Ch_T^{k+1} |u|_{k+1, T}, \\ |u - \pi_T u|_{1, T} \leq Ch_T^k |u|_{k+1, T}, \\ \dots \dots \dots \\ |u - \pi_T u|_{s, T} \leq Ch_T^{k+1-s} |u|_{k+1, T}$$

and so,

$$\|u - \pi_T u\|_{s,T} \leq \left(\sum_{T \in \mathcal{T}_h} \|u - \pi_T u\|_{s,T}^2 \right)^{1/2} \leq Ch^{k+1-s} \left(\sum_{T \in \mathcal{T}_h} |u|_{k+1,T}^2 \right)^{1/2} \leq Ch^{k+1-s} |u|_{k+1,\Omega},$$

where we noted $h = \max_{T \in \mathcal{T}_h} h_T$.

If we particularize the theorem for problems of first order and respectively second order and then for problems of third order and respectively fourth order on polygonal domains, we get.

Theorem 5. Let be (T, Σ, P) , $T \in \mathcal{T}_h$, $h \in \mathcal{H}$ a regular family of finite element of C^0 class, having $(\hat{T}, \hat{\Sigma}, \hat{P})$ as reference element. We suppose there is a whole $k \geq 1$ so that

$$P^k(\hat{T}) \subset \hat{P} \subset H^1(\hat{T}), \quad H^{k+1}(\hat{T}) \hookrightarrow C^0(\hat{T}).$$

If the solution u to variational problem of first or second order is in $H^{k+1}(\Omega)$ and u_h is the solution to the corresponding discrete problem, then there exists a constant C independent of h so that

$$(4.4) \quad \|u - u_h\|_{1,\Omega} \leq Ch^k |u|_{k+1,\Omega}, \quad h \in \mathcal{H}.$$

Theorem 6. Let be (T, Σ, P) , $T \in \mathcal{T}_h$, $h \in \mathcal{H}$ a regular family of finite element of class C^1 having $(\hat{T}, \hat{\Sigma}, \hat{P})$ as reference element. We suppose there is an integer $k \geq 2$ so that

$$P^k(\hat{T}) \subset \hat{P} \subset H^2(\hat{T}), \quad H^{k+1}(\hat{T}) \hookrightarrow C^1(\hat{T}).$$

If the solution u to the variational problem of third or fourth order is in $H^{k+1}(\Omega)$ and u_h is the solution to the corresponding discrete problem, then there exists a constant C independent of h so that

$$(4.5) \quad \|u - u_h\|_{2,\Omega} \leq Ch^{k-1} |u|_{k+1,\Omega}.$$

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REFERENCES

1. Ciarlet G. Ph., *The Finite Element Method for Elliptic Problems*. North Holland, 1980.
2. Diñcă G., *Metode variaționale și aplicații*. Ed. tehn., Buc., 1980.
3. Мартинюк А. Е., *Вариационные методы краевых задачи для слабо-эллиптических уравнений*. ДАН СССР, **129**, 1222—1225 (1959).
4. Petrila T., Gheorghiu C. I., *Metode de element finit și aplicații*. Ed. Acad. R.S.R., Buc., 1987.
5. Petrishin W. V., *Discret and Iterative Methods for Solution of Linear Operator Equations in Hilbert Spaces*. Trans. Amer. Math. Soc., **105**, 135—175 (1962).
6. Pletea A., *Metode de element finit pentru ecuații slab eliptice (ordin impar)*. Lucrările sesiunii jubiliare de comunicări științifice, Inst. Polit. Iași, 10—12 noiembrie 1988, pp. 189—192.

METODA ELEMENTULUI FINIT PENTRU ECUAȚII LINIARE SLAB ELIPTICE

(Rezumat)

Lucrarea este consacrată aplicării metodei elementului finit ecuațiilor slab eliptice liniare considerate pe un domeniu mărginit din $\mathbb{R}^2$.

THEORETICAL STUDIES ON THE EQUATIONS OF SEMI-INFINITE ELASTIC ROD — THE STATIONARY CASE

BY

RODICA LUCA

In this paper we will study the equation of semi-infinite elastic rod

$$(1) \quad \frac{d^4 u}{dx^4} - \beta^2 \frac{d^2 u}{dx^2} + A(x, u) = f(x), \quad x > 0, \quad \beta = \text{const.} \in \mathbb{R},$$

with the boundary-value conditions

$$(2) \quad \left\{ \begin{array}{l} -\frac{d^3 u}{dx^3}(0) + \beta^2 \frac{du}{dx}(0) \\ \frac{d^2 u}{dx^2}(0) \\ u \in H^4(\mathbb{R}_+). \end{array} \right\} \in l \left(\begin{array}{l} u(0) \\ \frac{du}{dx}(0) \end{array} \right).$$

We begin with the enumeration of some basic hypotheses that we shall use in the sequel :

(H.1) $l : D(l) \subset \mathbb{R}^2 \rightarrow 2^{\mathbb{R}^2}$ is a maximal monotone multifunction.

(H.2) 1° Functions $r \rightarrow A(x, r)$ are defined, continuous and monotone increasing on $\mathbb{R}$, for x a.e. on $\mathbb{R}_+$.

2° $A(\cdot, r) \in L^2(\mathbb{R}_+)$, $\forall r \in \mathbb{R}$.

3° $|A(x, r)| \leq C_1 |r| + C_2(x)$, $\forall r \in \mathbb{R}$, for x a.e. on $\mathbb{R}_+$, where $C_1 = \text{const.}$, $C_1 \geq 0$ and $C_2 \in L^2(\mathbb{R}_+)$.

(H.3) $f \in L^2(\mathbb{R}_+)$.

We will denote by $\|\cdot\|_U$ the norm in the normed linear space U and by $\langle \cdot, \cdot \rangle_H$ the scalar product in the Hilbert space H . By $\begin{bmatrix} x \\ y \end{bmatrix}$ we denote an arbitrary element in the space $\mathbb{R}^2$ ($x, y \in \mathbb{R}$).

The condition (2) given above is a very general condition in certain particular cases, containing boundary conditions well known to us. In this sense we give some examples :

1) For $l = \partial\varphi$, where $\varphi: \mathbb{R}^2 \rightarrow]-\infty, \infty]$ is defined by

$$\varphi\left(\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}\right) = \begin{cases} 0, & \text{if } r_1 = a \text{ and } r_2 = b \\ +\infty, & \text{in rest,} \end{cases} \quad \forall \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \in \mathbb{R}^2$$

(is showing easily that φ is a proper, lower-semicontinuous, convex function) the condition (2) becomes

$$u(0) = a, \quad \frac{du}{dx}(0) = b, \quad u \in H^4(\mathbb{R}_+).$$

These conditions appear, for example, in the case of a rod with thin walls and open section that can buckle, as a result of a twist and bending combined load. In this case the rod is embedded at an extremity, representing the twist angle of rod ($a=b=0$).

2) If $l = \partial j$, where $j: \mathbb{R}^2 \rightarrow]-\infty, +\infty]$ is defined by

$$j\left(\begin{bmatrix} r_1 \\ r_2 \end{bmatrix}\right) = \begin{cases} r_2 b & \text{if } r_1 = a \\ +\infty, & \text{in rest,} \end{cases} \quad \forall \begin{bmatrix} r_1 \\ r_2 \end{bmatrix} \in \mathbb{R}^2$$

(j is a proper, lower-semicontinuous, convex function) thus the condition (2) becomes

$$u(0) = a, \quad \frac{d^2 u}{dx^2}(0) = b, \quad u \in H^4(\mathbb{R}_+),$$

that represent boundary conditions associated to a rod of the same type as in (1), and at this time, the rod being simply supported so that their ends are free to displace and rotate around of y and z axes (the principal axes that pass through the centre of gravity of section), but they can not rotate around of x axis and neither can bend on direction y or z ($a=b=0$).

3) For $l = \partial\varphi$, where $\varphi: \mathbb{R}^2 \rightarrow]-\infty, +\infty]$ is defined by

$$\varphi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = ax_1 + bx_2, \quad \forall \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2,$$

with a and $b \in \mathbb{R}$ (immediately is verified that φ is a proper, lower-semicontinuous, convex function), the condition (2) lead us to

$$\frac{d^2 u}{dx^2}(0) = b, \quad -\frac{d^3 u}{dx^3}(0) + \beta^2 \frac{du}{dx}(0) = a, \quad u \in H^4(\mathbb{R}_+).$$

4) Because a positive matrix is symmetric if and only if the operator defined by this matrix is subdifferential, we obtain a condition differing from that above mentioned considering that

$$l(u) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} u, \quad \forall u \in \mathbb{R}^2, \quad c \neq b,$$

where the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is positive. In this case we obtain the following conditions

$$-\frac{d^3 u}{dx^3}(0) + \beta^2 \frac{du}{dx}(0) = au(0) + b \frac{du}{dx}(0), \quad \frac{d^2 u}{dx^2}(0) = cu(0) + d \frac{du}{dx}(0), \quad u \in H^4(\mathbb{R}_+).$$

We will use the notation u' for designate the first derivative of the function u (in the sense of distribution). In the following X will be the space $L^2(\mathbb{R}_+)$.

We define the operator $T: D(T) \subset X \rightarrow X$ by

$$Tu = u'''' - \beta^2 u'', \quad \forall u \in D(T),$$

$$(3) \quad D(T) = \left\{ u \in H^4(\mathbb{R}_+); \begin{bmatrix} -u''''(0) + \beta^2 u'(0) \\ u''(0) \end{bmatrix} \in l \left(\begin{bmatrix} u(0) \\ u'(0) \end{bmatrix} \right) \right\}.$$

The operator T has the property that $D(T) \neq \emptyset$; moreover $D(T)$ is a dense set in X . For proving this, we consider the function

$$u_0(t) = At^3e^{-t} + Bt^2e^{-t} + Cte^{-t} + De^{-t}.$$

We observe that $u_0 \in C^\infty(\mathbb{R}_+)$ and we determine the constants A, B, C and D so that

$$u_0(0) = c, \quad u_0'(0) = d, \quad -u_0''''(0) + \beta^2 u_0'(0) = a \quad \text{and} \quad u_0''(0) = b,$$

where $\begin{bmatrix} a \\ b \end{bmatrix} \in l \left(\begin{bmatrix} c \\ d \end{bmatrix} \right)$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ is a fixed element of $\mathbb{R}^2$.

Computing the derivatives of function u_0 in the point $t=0$ and setting the above conditions, we obtain

$$A = \frac{1}{6}(-a + 3b + c + (3 + \beta^2)d), \quad B = \frac{1}{2}(b + c + 2d), \quad C = c + d, \quad D = c.$$

Because we have the following inclusion

$$u_0 + C_0^\infty(\mathbb{R}_+) \subset D(T) \subset X$$

and $C_0^\infty(\mathbb{R}_+)$ is a dense set in X and hence the translation of this set with u_0 has the same property, it follows that $D(T)$ is a dense set in X .

Lemma 1. *In hypothesis (H. 1) the operator T defined in (3) is maximal monotone.*

For the proof see [4].

In the following let consider the problem

$$(4) \quad u'''' - \beta^2 u'' + a_0(x)u = p(x),$$

$$(5) \quad \begin{bmatrix} -u''''(0) + \beta^2 u'(0) \\ u''(0) \end{bmatrix} \in l \left(\begin{bmatrix} u(0) \\ u'(0) \end{bmatrix} \right), \quad u \in H^4(\mathbb{R}_+),$$

where

$$(6) \quad a_0 \in L^\infty(\mathbb{R}_+), \quad a_0(x) \geq k_0 > 0, \quad \text{a.e. on } \mathbb{R}_+.$$

Proposition 1. *In hypotheses (H. 1) and (6) the problem (4)–(5) has solution for any $p \in X$.*

Proof. Let be the operators $P: D(P) = D(T) \subset X \rightarrow X$, $Pu = a_0(x)u$ and $Q: D(Q) = D(T) \subset X \rightarrow X$, $Q = T + P$. The operator Q is a strong monotone operator, since

$$\begin{aligned} \langle Qu - Qu_1, u - u_1 \rangle_X &\geq \langle Pu - Pu_1, u - u_1 \rangle_X = \int_0^\infty a_0(x) |u - u_1|^2 dx \geq \\ &\geq k_0 \int_0^\infty |u - u_1|^2 dx = k_0 \|u - u_1\|_X^2, \quad \forall u, u_1 \in D(Q). \end{aligned}$$

Because Q is linear, too, it follows that the operator Q is coercive. With the observation that the operator Q is maximal monotone and X is a Hilbert space we conclude that $R(Q)=X$, hence the problem (4)–(5) has solution for any $p \in X$. Q.E.D.

Observation. For $a_0 \equiv 0$ the above result is not maintained. Indeed, for $p(x) \equiv 0$ and $l(w) = [\frac{1}{-1}]$, $\forall w \in \mathbb{R}^2$ one should prove easily that the problem (4)–(5) has no solutions.

Proposition 2. If $l = \partial j$, where $j: \mathbb{R}^2 \rightarrow]-\infty, +\infty]$ is a proper, convex and lower-semicontinuous function, then $T = \partial \varphi$, where $\varphi: X \rightarrow]-\infty, +\infty]$ defined by

$$(\varphi u) = \frac{1}{2} \|u''\|_X^2 + \frac{1}{2} \beta^2 \|u'\|_X^2 + j\left(\begin{bmatrix} u(0) \\ u'(0) \end{bmatrix}\right), \quad \forall u \in X$$

is a proper, convex and lower-semicontinuous function.

Proof. Firstly, we observe that the function φ is proper, that is, it exists $u \in X$ so that $\varphi(u) < \infty$, namely $u(t) = (b+a)te^{-t} + ae^{-t}$ with $j(\begin{bmatrix} b \\ a \end{bmatrix}) < \infty$. Because the norm function and j are convex it follows that φ is a convex function. Now, we show that the function φ is lower-semicontinuous, that is – equivalently – the level sets are closed.

Let be $\lambda \in \mathbb{R}$ fixed, $A = \{u \in X; \varphi(u) \leq \lambda\}$ the level set corresponding to λ and $\{u_n\} \subset A$, $u_n \rightarrow u$ for $n \rightarrow \infty$, that is $u_n \in X$, $\forall n \in \mathbb{N}$ and $\varphi(u_n) \leq \lambda$ or

$$(7) \quad \frac{1}{2} \|u_n''\|_X^2 + \frac{1}{2} \beta^2 \|u_n'\|_X^2 + j\left(\begin{bmatrix} u_n(0) \\ u_n'(0) \end{bmatrix}\right) \leq \lambda, \quad \forall n \in \mathbb{N}.$$

Since j is bounded from below by an affine function it follows that exist $(x_0, y_0) \in \mathbb{R}^2$ and $\lambda_0 \in \mathbb{R}$ so that

$$j\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) \geq x_0 u + y_0 v + \lambda_0, \quad \forall \begin{bmatrix} u \\ v \end{bmatrix} \in \mathbb{R}^2.$$

Using (7) we get

$$\frac{1}{2} \|u_n''\|_X^2 + \frac{1}{2} \beta^2 \|u_n'\|_X^2 + x_0 u_n(0) + y_0 u_n'(0) \leq \bar{\lambda}, \quad \bar{\lambda} = \lambda - \lambda_0 \in \mathbb{R}$$

or

$$(8) \quad \frac{1}{2} \|u_n''\|_X^2 + \frac{1}{2} \beta^2 \|u_n'\|_X^2 \leq \bar{\lambda} + |x_0| |u_n(0)| + |y_0| |u_n'(0)|.$$

But

$$\begin{aligned} |u_n(x)| &= \left| \int_0^1 [su_n'(s) + u_n(s)] ds - \int_x^1 u_n'(s) ds \right| \leq \int_0^1 |u_n'(s)| ds + \int_0^1 |u_n(s)| ds + \\ &+ \int_x^1 |u_n'(s)| ds \leq 2 \int_0^1 |u_n'(s)| ds + \int_0^1 |u_n(s)| ds \leq 2 \sqrt{\int_0^1 |u_n'(s)|^2 ds} + \sqrt{\int_0^1 |u_n(s)|^2 ds}, \\ &\quad \forall x \in [0, 1]. \end{aligned}$$

This yields

$$(9) \quad \|u_n\|_{C([0,1])} \leq 2 \|u_n'\|_X + \|u_n\|_X$$

and, similarly, for u_n'

$$(10) \quad \|u'_n\|_{C([0,1])} \leq 2\|u''_n\|_X + \|u'_n\|_X.$$

From (8), (9) and (10) we obtain

$$\frac{1}{2}\|u''_n\|_X^2 + \frac{1}{2}\beta^2\|u'_n\|_X^2 \leq \bar{\lambda} + (2|\bar{x}_0| + |y_0|) + \|u'_n\|_X + |x_0| \|u_n\|_X + 2|y_0| \|u''_n\|_X,$$

or

$$\|u''_n\|_X^2 + \|u'_n\|_X^2 \leq C_3 + C_4(\|u'_n\|_X + \|u''_n\|_X),$$

where C_3 and C_4 are constants, with the observation that the sequence $\{u_n\}$ is bounded in X .

From the above inequalities it follows that the sequences $\{u'_n\}$ and $\{u''_n\}$ are bounded in X . Hence

$$\begin{cases} u'_n \rightarrow u' & \text{weak in } X \text{ for } n \rightarrow \infty, \\ u''_n \rightarrow u'' & \text{weak in } X \text{ for } n \rightarrow \infty. \end{cases}$$

Arguing as in the demonstration of Lemma 1 (see [4]) it follows that the sequences $\{u_n\}$, $\{u'_n\}$ restricted to $[0, 1]$ are relatively compact in $C([0, 1])$, hence

$$\begin{cases} u_n(0) \rightarrow u(0) & \text{for } n \rightarrow \infty. \\ u'_n(0) \rightarrow u'(0) & \text{for } n \rightarrow \infty. \end{cases}$$

Because j is lower-semicontinuous it follows that

$$\liminf_{n \rightarrow \infty} j \left(\begin{bmatrix} u_n(0) \\ u'_n(0) \end{bmatrix} \right) = \begin{bmatrix} u(0) \\ u'(0) \end{bmatrix}.$$

By letting $n \rightarrow \infty$ in the inequality (7) we obtain

$$\frac{1}{2}\|u''\|_X^2 + \frac{1}{2}\beta^2\|u'\|_X^2 + j \left(\begin{bmatrix} u(0) \\ u'(0) \end{bmatrix} \right) \leq \lambda,$$

i.e. $u \in A$, hence the function φ is lower-semicontinuous.

Now we show that $T = \partial\varphi$. For this it suffices to demonstrate the inclusion $T \subset \partial\varphi$, because T being a maximal monotone operator, it will result $T = \partial\varphi$.

Let be $u \in D(T)$. We must show that $Tu \in \partial\varphi(u)$ i.e.

$$(11) \quad \varphi(u) - \varphi(v) \leq \langle Tu, u - v \rangle_X, \quad \forall v \in D(\varphi).$$

We have

$$\begin{aligned} \langle Tu, u - v \rangle_X &= \langle u'''' - \beta^2 u'', u - v \rangle_X = \int_0^\infty (u'''' - \beta^2 u'')(u - v) dx = \\ &= (u'''' - \beta^2 u'')(u - v) \Big|_0^\infty - \int_0^\infty (u'''' - \beta^2 u'')(u' - v') dx = -[u''''(0) - \beta^2 u''(0)][u(0) - \\ &- v(0)] - u''(u' - v') \Big|_0^\infty + \int_0^\infty u''(u'' - v'') dx + \beta^2 \int_0^\infty u'(u' - v') dx = -[u''''(0) - \beta^2 u''(0)]. \end{aligned}$$

$$\begin{aligned}
& \cdot [u(0) - v(0)] + u''(0)[u'(0) - v'(0)] + \int_0^\infty u''(u'' - v'') dx + \beta^2 \int_0^\infty u'(u' - v') dx \geq \\
& \geq j\left(\begin{bmatrix} u(0) \\ u'(0) \end{bmatrix}\right) - j\left(\begin{bmatrix} v(0) \\ v'(0) \end{bmatrix}\right) + \int_0^\infty u''(u'' - v'') dx + \beta^2 \int_0^\infty u'(u' - v') dx \geq j\left(\begin{bmatrix} u(0) \\ u'(0) \end{bmatrix}\right) - \\
& - j\left(\begin{bmatrix} v(0) \\ v'(0) \end{bmatrix}\right) + \int_0^\infty |u''|^2 dx - \frac{1}{2} \int_0^\infty |u''|^2 dx - \frac{1}{2} \int_0^\infty |v''|^2 dx + \\
& + \beta^2 \int_0^\infty |u'|^2 dx - \frac{1}{2} \beta^2 \int_0^\infty |u'|^2 dx - \frac{1}{2} \beta^2 \int_0^\infty |v'|^2 dx = j\left(\begin{bmatrix} u(0) \\ u'(0) \end{bmatrix}\right) - j\left(\begin{bmatrix} v(0) \\ v'(0) \end{bmatrix}\right) + \\
& + \frac{1}{2} \|u''\|_X^2 + \frac{1}{2} \beta^2 \|u'\|_X^2 - \frac{1}{2} \|v''\|_X^2 - \frac{1}{2} \beta^2 \|v'\|_X^2 = \varphi(u) - \varphi(v), \quad \forall v \in D(\varphi).
\end{aligned}$$

This completes the proof. Q.E.D.

With the above notations, let consider the following function $\tilde{\varphi}: X \rightarrow]-\infty, +\infty]$ defined by

$$(12) \quad \tilde{\varphi}(u) = \frac{1}{2} \|u''\|_X^2 + \frac{1}{2} \beta^2 \|u'\|_X^2 + j\left(\begin{bmatrix} u(0) \\ u'(0) \end{bmatrix}\right) + \frac{1}{2} \int_0^\infty a_0(x) |u|^2 dx - \int_0^\infty p(x) u(x) dx,$$

$\forall u \in X.$

Proposition 3. *In the hypotheses of Proposition 2, u is a solution for the problem (4)–(5) if and only $u \in D(T)$ is a point of minimum for the function $\tilde{\varphi}$.*

Proof. Because

$$\tilde{\varphi}(u) = \varphi(u) + \frac{1}{2} \int_0^\infty a_0(x) |u|^2 dx - \int_0^\infty p(x) u(x) dx, \quad \forall u \in X,$$

from demonstration of the proposition (2) we deduce that $\tilde{\varphi}$ is a proper, lower-semicontinuous, convex function.

We are knowing that $u \in D(T)$ is a point of minimum for the function $\tilde{\varphi}$ if and only $\partial \tilde{\varphi}(u) = 0$. But, in according to the proposition (2)

$$\partial \tilde{\varphi}(u) = \partial \varphi(u) + a_0(x)u - p(x) = Tu + a_0(x)u - p(x), \quad \forall u \in D(T),$$

from which results that $\partial \tilde{\varphi}(u) = 0$, therefore $u \in D(T)$ is a solution for the problem (4)–(5). Q.E.D.

Now, we define the operator $\mathcal{A}: D(\mathcal{A}) = D(T) \subset X \rightarrow X$ by

$$(13) \quad \mathcal{A}u = Tu + A(x, u), \quad \forall u \in D(\mathcal{A}).$$

Lemma 2. *In the hypotheses (H.1) and (H.2) the operator $\mathcal{A}$ is maximal monoton.*

P r o o f . Firstly, we show that the operator $\mathcal{A}$ is monotone. For this it suffices to demonstrate that the operator $A(x, \cdot)$ is monotone. According to the hypotheses (H.2) 1°, we have

$$\langle A(x, u_1) - A(x, u_2), u_1 - u_2 \rangle_X = \int_0^\infty [A(x, u_1(x)) - A(x, u_2(x))] [u_1(x) - u_2(x)] dx \geq 0,$$

$\forall u_1, u_2 \in D(\mathcal{A})$, that is $A(x, \cdot)$ is monotone.

We observe that the operator $u \rightarrow A(\cdot, u(\cdot))$ is all over defined and monotone; moreover, this operator in hypotheses (H.2) 2°–3° is bounded and continuous (see [5], p. 166 with the corresponding adaptations). Because T is a maximal monotone operator, we conclude that the operator $\mathcal{A}$ is maximal monotone, too. Q.E.D.

The main result is the

T h e o r e m . *If the function A satisfies the condition*

$$(14) \quad (A(x, r_1) - A(x, r_2))(r_1 - r_2) \geq \omega |r_1 - r_2|^2, \quad \omega = \text{const} \in \mathbb{R}_+^*, \\ \forall r_1, r_2 \in \mathbb{R}, \text{ for } x \text{ a.e. on } \mathbb{R}_+,$$

then in hypotheses (H.1), (H.2) and (H.3) the problem (1)–(2) has solution.

To prove the theorem it suffices to show that the operator $\mathcal{A}$ —about we know from Lemma 2 that is maximal monotone—is coercive. Then will result that $R(\mathcal{A}) = X$, that is, for any $f \in X$ the problem (1)–(2) has solution. But using the condition (14) and the fact that T is a linear operator it follows that $\mathcal{A}$ is a coercive operator. Q.E.D.

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REFERENCES

1. Adams R. A., *Sobolev Spaces*. Academic Press, New York, 1975.
2. Barbu V., *Nonlinear Semigroups and Differential Equations in Banach Spaces*. Noordhoff, Leyden, 1976.
3. Lions J. I., Magenes E., *Problèmes aux limites non homogènes et applications*. T. 1, Dunod, Paris, 1968.
4. Luca R., *Probleme calitative pentru ecuații neliniare de evoluție în spații Hilbert*. Lucrare de diplomă, Facultatea de Matematică, Iași, 1984.
5. Luca R., *On a Maximal Monotone Operator* (to appear).
6. Pascali D., Sburlan S., *Nonlinear Mappings of Monotone Type*. Ed. Acad., Buc., 1978.
7. Timoshenko S. P., Gere J. M., *Teoria stabilității elastice*. Ed. tehn., Buc., 1967.

STUDII TEORETICE ASUPRA ECUAȚIEI VERGELEI ELASTICE SEMI-INFINITE — CAZUL STAȚIONAR

(Rezumat)

Se studiază ecuația (1) cu condițiile la limită (2).

CONTINUOUS SELECTIONS FOR MULTIFUNCTIONS AND THE CAUCHY PROBLEM FOR MULTIVALUED EQUATIONS

BY

GEORGETA TEODORU

1. Introduction. In this paper we prove an existence theorem of a continuous selection for a multifunction F defined in a compact subset of $\mathbb{R}^{n+2}$ and taking compact nonempty values not necessarily convex. The theorem establishes the existence of a continuous selection for each of the functions $(x, y) \rightarrow F(x, y, z(x, y))$, with respect to a given family $\{x, (y, z)\}$ of continuous functions $(x, y) \rightarrow z(x, y)$. This result is stronger than Theorem 1 [5] and it is analogous of [1]. As corollary, we obtain the existence of a solution for the Cauchy problem, associated with the multivalued hyperbolic equation $\partial^2 z / \partial x \partial y \in F(x, y, z)$ [4].

2. Preliminary Results [5]. Let be the multifunction $F: D \times B \rightarrow \text{comp} X$, $D = [0, a] \times [0, b]$, B is the closed ball centered in origin of $\mathbb{R}^n$ and with radius $c = M_1 + Mab$, M_1 given by (2.4), M given by (2.6), X is the closed ball centered in origin of $\mathbb{R}^n$ and with radius M . Obviously, X is a compact space for the metric d induced on X by the norm of $\mathbb{R}^n$. Let H be the Hausdorff-Pompeiu metric [3] on $\text{comp} X$ induced by d . Then $\text{comp} X$ is a compact metric space for H .

Let $\mathcal{C}(D; \mathbb{R}^n)$ be the Banach space of continuous functions from D into $\mathbb{R}^n$ and $\mathcal{L}^1(D; \mathbb{R}^n)$ the Banach space of equivalence classes of Lebesgue integrable functions on D and valued in $\mathbb{R}^n$. Consider the following hypotheses;

(H₀) The curve $\gamma: y = \varphi(x)$, $0 \leq x \leq a$, is defined by the function φ , $\varphi \in C^1([0, a]; \mathbb{R})$, $\varphi' \in AC([0, a]; \mathbb{R})$ satisfying the relations

$$(2.1) \quad \varphi(0) = 0, \quad \varphi(a) = b, \quad \varphi'(x) > 0, \quad 0 \leq x \leq a.$$

We remark that the equation of the curve γ can be written under the form $\gamma: x = \psi(y)$, $0 \leq y \leq b$, $\psi = \varphi^{-1}$, submitted to the conditions

$$(2.2) \quad \psi(0) = 0, \quad \psi(b) = a, \quad \psi'(y) > 0, \quad 0 \leq y \leq b.$$

(H₁) The functions R, P, R' belong to $AC([0, a]; \mathbb{R}^n)$, where $AC([\alpha_1, \alpha_2]; \mathbb{R}^n)$ is the space of absolutely continuous functions $f: [\alpha_1, \alpha_2] \rightarrow \mathbb{R}^n$ normed by

$$\|f\| = \sup_{t \in [\alpha_1, \alpha_2]} \|f(t)\| + \int_{\alpha_1}^{\alpha_2} \|f'(t)\| dt,$$

and the function $\alpha: D \rightarrow \mathbb{R}^n$, defined by

$$(2.3) \quad \alpha(x, y) = R(x) - \int_y^{\varphi(x)} Q(\psi(v)) dv, \quad (x, y) \in D,$$

is bounded; there exists $M_1 > 0$ such that

$$(2.4) \quad \|\alpha(x, y)\| \leq M_1, \quad (x, y) \in D.$$

In (2.3), $Q: [0, a] \rightarrow \mathbb{R}^n$ is the function defined by

$$(2.5) \quad Q(x) = \frac{1}{\varphi'(x)} [R'(x) - P(x)], \quad 0 \leq x \leq a.$$

It follows that α is absolutely continuous function in the Carathéodory sense [2, §565–§568], $\alpha \in C^*(D; \mathbb{R}^n)$.

Let $\mathcal{X}$ be the set of absolutely continuous functions [2], $z: D \rightarrow \mathbb{R}^n$, $z \in C^*(D; \mathbb{R}^n)$ which satisfy (2.3) ... (2.7), where

$$(2.6) \quad \left\| \frac{\partial^2 z(x, y)}{\partial x \partial y} \right\| \leq M, \quad \text{a.e. } (x, y) \in D,$$

and

$$(2.7) \quad z(x, \varphi(x)) = R(x), \quad \frac{\partial z}{\partial x}(x, \varphi(x)) = P(x), \quad 0 \leq x \leq a.$$

Proposition 1. *The set $\mathcal{X}$ is a nonempty compact and convex subset of $\mathcal{C}(D; \mathbb{R}^n)$.*

Proof. The relation $z \in \mathcal{X}$ implies $z \in \mathcal{C}(D; \mathbb{R}^n)$. We observe that $\partial^2 z / \partial x \partial y$ exists a.e. $(x, y) \in D$, as $z \in C^*(D; \mathbb{R}^n)$, [2]. Let $N(x, y)$, $N'(x) = \psi(y)$, $N''(x, \varphi(x))$, with $y \leq \varphi(x)$, three points in D and the triangle

$$T(x, y) = \{(u, v) | \psi(v) \leq u \leq x, y \leq v \leq \varphi(x)\}, \quad (x, y) \in D.$$

Integrating $\partial^2 z / \partial x \partial y$ in $T(x, y)$ and using (2.3) and (2.7) we obtain

$$(2.8) \quad \begin{aligned} z(x, y) &= R(x) - \int_y^{\varphi(x)} Q(\psi(v)) dv - \int_y^{\varphi(x)} dv \int_{\psi(v)}^x \frac{\partial^2 z(u, v)}{\partial u \partial v} du = \\ &= \alpha(x, y) - \iint_{T(x, y)} \frac{\partial^2 z(u, v)}{\partial u \partial v} dudv, \quad (x, y) \in D. \end{aligned}$$

The compactness of $\mathcal{X}$ follows from (2.8) and the Arzela-Ascoli Theorem and its convexity is obvious.

Remark. The relation $z \in \mathcal{X}$ implies $(x, y, z(x, y)) \in D \times B$ for every $(x, y) \in D$. Because each $z \in \mathcal{X}$ generates a multifunction $(x, y) \rightarrow F(x, y, z(x, y))$ from D into $\text{comp} X$, we shall denote this mapping by $G(z): D \rightarrow \text{comp} X$, where

$$(2.9) \quad G(z)(x, y) = F(x, y, z(x, y)), \quad (x, y) \in D.$$

3. Continuous Selections. The Continuous Case Refined. We prove the following result, analogous to Lemma 3, [1].

Lemma. Let $A:D \rightarrow \text{comp} X$ a continuous multifunction and $v:D \rightarrow \mathbb{R}^n$ a piecewise constant mapping such that $d(v(x, y), A(x, y)) < \rho$ for every $(x, y) \in D$. Then, for every $\varepsilon > 0$, there exists a piecewise constant mapping $w:D \rightarrow \mathbb{R}^n$ such that $d(v(x, y), w(x, y)) < \varepsilon$ and $d(w(x, y), A(x, y)) < \rho$ for every $(x, y) \in D$.

Proof. Indeed, given $\varepsilon > 0$, we can choose a partition $(D_{ij})_{1 \leq i \leq m, 1 \leq j \leq n}$ of $J = [0, a] \times [0, b]$ consisting of intervals $D_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$, such that $v|_{D_{ij}} = z_{ij}$ and $H(A(x, y), A(x', y')) < \varepsilon$ for any $(x, y), (x', y')$ in D_{ij} . Then, for each (i, j) there exists a point $\zeta_{ij} \in A(x_{i-1}, y_{j-1})$ such that $d(v(x_{i-1}, y_{j-1}), \zeta_{ij}) < \rho$ and $d(\zeta_{ij}, A(x, y)) < \varepsilon$ for every $(x, y) \in D_{ij}$. We define the mapping $w:D \rightarrow \mathbb{R}^n$ as follows: $w|_{D_{ij}} = \zeta_{ij}$ for each (i, j) , $w(a, y) = \lim_{x \rightarrow a} w(x, y)$, $w(x, b) = \lim_{y \rightarrow b} w(x, y)$. The mapping w has the required properties. Obviously, if $(x, y) \in J$, then $(x, y) \in D_{ij}$ for an unique D_{ij} , such that $w(x, y) = \zeta_{ij}$ and $v(x, y) = z_{ij} = v(x_{i-1}, y_{j-1})$, and consequently $d(v(x, y), w(x, y)) = d(v(x_{i-1}, y_{j-1}), \zeta_{ij}) < \rho$ and $d(w(x, y), A(x, y)) = d(\zeta_{ij}, A(x, y)) < \varepsilon$. By continuity, these inequalities are also true and for $x=a, y=b$.

Theorem. Let $F:D \times B \rightarrow \text{comp} X$ be a continuous multifunction. Then there exists a continuous mapping $g:\mathcal{K} \rightarrow \mathcal{L}^1(D; \mathbb{R}^n)$ such that, for every $z \in \mathcal{K}$, $g(z)$ is a regulated mapping in D and $g(z)(x, y) \in G(z)(x, y)$ for every $(x, y) \in D$.

Proof. We shall construct, for every $n \geq 1$, a continuous mapping $g^n:\mathcal{K} \rightarrow \mathcal{L}^1(D; \mathbb{R}^n)$ such that, for every $z \in \mathcal{K}$, $g^n(z)$ is a piecewise constant mapping of D into X which satisfies, at every $(x, y) \in D$,

$$(3.1) \quad d(g^n(z)(x, y), G(z)(x, y)) < 2^{-n},$$

$$(3.2) \quad \|g^{n+1}(z)(x, y) - g^n(z)(x, y)\| < 2^{-n-1}.$$

It follows that for every $z \in \mathcal{K}$, the sequence $(g^n(z))$ converges uniformly in D to a mapping $g(z)$ of D into X that is regulated in D and satisfies $g(z)(x, y) \in G(z)(x, y)$ at every $(x, y) \in D$. Indeed, since the convergence is uniform in D and each g^n is continuous in $\mathcal{K}$, g will be a continuous mapping of $\mathcal{K}$ into $\mathcal{L}^1(D; \mathbb{R}^n)$ and this will prove the statement. The construction will be made in two stages. First choose a decreasing null sequence (Δn) of positive constants such that, for every $n \geq 1$,

$$(3.3) \quad H(F(x, y, z), F(\xi, \eta, \zeta)) < 2^{-n-3},$$

for any points $(x, y, z), (\xi, \eta, \zeta)$ in $D \times B$ with $\|(x, y) - (\xi, \eta)\| < \Delta n, \|z - \zeta\| < \Delta n$. This is possible, because F is uniformly continuous in D . We select for every $n \geq 1$, a finite open covering $(U_k^n)_{1 \leq k \leq N(n)}$ of the compact space $\mathcal{K}$ such that

$$\text{diam } U_k^n < \Delta n, \quad 1 \leq k \leq N(n).$$

Let $(p_k^n)_{1 \leq k \leq N(n)}$ be a continuous partition of unity subordinate to $(U_k^n)_{1 \leq k \leq N(n)}$. We denote $N(n) = \overline{N_1(n)N_2(n)}$ and $p_k^n(z) = q_i^n(z)r_j^n(z), i = \overline{1, N_1(n)}, j = \overline{1, N_2(n)}$. The functions $p_i^n:\mathcal{K} \rightarrow \mathbb{R}$ satisfy the properties:

$$a) \quad 0 \leq p_{ij}^n(z) \leq 1 \quad \text{for } z \in \mathcal{K}, i = \overline{1, N_1(n)}, j = \overline{1, N_2(n)},$$

$$b) \quad p_{ij}^n(z) = 0 \quad \text{if } z \notin U_{ij}^n, \quad i = \overline{1, N_1(n)}, j = \overline{1, N_2(n)},$$

$$c) \sum_{i=1}^{N_1(n)} \sum_{j=1}^{N_2(n)} p_{ij}^n(z) = \sum_{i=1}^{N_1(n)} \sum_{j=1}^{N_2(n)} q_i^n(z) r_j^n(z) = 1, \quad \text{for } z \in \mathcal{X}.$$

We denote

$$W_k^n = \{z \in U_k^n \mid p_k^n(z) > 0\}, \quad 1 \leq k \leq N(n).$$

Then, for every $n \geq 1$ and every vector index $l = (l_1, l_2, \dots, l_n)$ such that $1 \leq l_v \leq N(v)$, $\bigcap_{v=1}^n W_{l_v}^\nu \neq \emptyset$, there exists a piecewise constant mapping $v_l^n: D \rightarrow X$ and a point $z_l^n \in \bigcap_{v=1}^n W_{l_v}^\nu$ such that, at every $(x, y) \in D$.

$$(3.4) \quad d(v_l^n(x, y), G(z_l^n)(x, y)) < 2^{-n-1}.$$

This assertion is obviously true for $n=1$. Suppose that it is true for $n = 1, 2, \dots, p$. If $l = (l_1, l_2, \dots, l_n)$ is such that (3.4) holds for $n=p$, we can use Lema and construct, for every integer s such that

$$1 \leq s \leq N(p+1), \quad \bigcap_{v=1}^p W_{l_v}^\nu \cap W_s^{p+1} \neq \emptyset$$

a piecewise constant mapping $v_{(l,s)}^{p+1}: D \rightarrow X$ which satisfies, at every $(x, y) \in D$,

$$(3.5) \quad d(v_{(l,s)}^{p+1}(x, y), G(z_l^p)(x, y)) < 2^{-p-3},$$

$$(3.6) \quad \|v_{(l,s)}^{p+1}(x, y) - v_l^p(x, y)\| < 2^{-p-1}.$$

Remark. For any n -vector index $l = (l_1, l_2, \dots, l_n)$ and integer s , we denote by (l, s) the $n+1$ -vector index $(l_1, l_2, \dots, l_n, s)$.

Thus, if we fix a point

$$z_{(l,s)}^{p+1} \in \bigcap_{v=1}^p W_{l_v}^\nu \cap W_s^{p+1},$$

we deduce for every $(x, y) \in D$,

$$\begin{aligned} d(v_{(l,s)}^{p+1}(x, y), G(z_{(l,s)}^{p+1}(x, y))) &\leq d(v_{(l,s)}^{p+1}(x, y), G(z_l^p)(x, y)) + \\ &+ H(G(z_l^p)(x, y), G(z_{(l,s)}^{p+1}(x, y))) < 2^{-p-3} + 2^{-p-3} = 2^{-(p+1)-1}. \end{aligned}$$

Hence, the assertion is true for $n=p+1$ and consequently, by induction, for every $n \geq 1$.

We next define, for every $z \in \mathcal{X}$, a sequence of finite partitions of the interval J as follows; given $z \in \mathcal{X}$, we successively construct, for every $n \geq 1$ and every $2n$ -vector index $l = (l_1, l_2, \dots, l_n)$, $l = l^1 \times l^2$, $l^1 = (l_1^1, l_2^1, \dots, l_n^1)$, $l^2 = (l_1^2, l_2^2, \dots, l_n^2)$, $1 \leq l_v^1 \leq N_1(v)$, $1 \leq l_v^2 \leq N_2(v)$, $1 \leq v \leq n$, an interval $J_l^n(z) \subset J$, such that

$$(3.7) \quad J = \bigcup_{\substack{1 \leq i \leq N_i(l^1) \\ 1 \leq j \leq N_i(l^1)}} J_i^1(z)$$

and

$$(3.8) \quad J_l^n(z) = \bigcup_{1 \leq s \leq N(n+1)} J_{(l,s)}^{n+1}(z), \quad n \geq 1.$$

Indeed, let

$$\begin{cases} x_0^1(z) = 0, \\ x_i^1(z) = x_{i-1}^1(z) + a q_i^1(z) \sum_{j=1}^{N_1(1)} r_j^1(z), \quad i = \overline{1, N_1(1)}, \end{cases}$$

and

$$\begin{cases} y_0^1(z) = 0, \\ y_j^1(z) = y_{j-1}^1(z) + b r_j^1(z) \sum_{i=1}^{N_1(1)} q_i^1(z), \quad j = \overline{1, N_2(1)}. \end{cases}$$

We denote

$$J_{ij}^1(z) = [x_{i-1}^1(z), x_i^1(z)[x[y_{j-1}^1(z), y_j^1(z)[$$

for each $i = \overline{1, N_1(1)}$, $j = \overline{1, N_2(1)}$. Then, obviously, $J_{ij}^1(z)$ is nonempty if and only if $z \in W_{ij}^1$, but (3.7) holds whatever $z \in \mathcal{K}$ because $(p_{ij}^1)_{1 \leq i \leq N_1(1), 1 \leq j \leq N_2(1)}$ is a partition of unity. More generally, if $l = (l_1, l_2, \dots, l_n)$ is an $2n$ -vector, index with $1 \leq l_v \leq N(v)$, $l = l^1 \times l^2$, $l^1 = (l_1^1, l_2^1, \dots, l_n^1)$, $l^2 = (l_1^2, l_2^2, \dots, l_n^2)$, $1 \leq l_v^1 \leq N_1(v)$, $1 \leq l_v^2 \leq N_2(v)$, $1 \leq v \leq n$, for which $J_l^n(z)$ has been constructed, let

$$\begin{cases} x_{(l^1, 0)}^n(z) = x_{(l_1^1, l_2^1, \dots, l_n^1 - 1)}^n(z), \\ x_{(l^1, s^1)}^{n+1}(z) = x_{(l^1, s^1 - 1)}^{n+1}(z) + \left(a \prod_{v=1}^n q_{l_v^1}^v(z) \sum_{j=1}^{N_1(n)} r_j(z) \right) q_{s^1}^{n+1}(z) \end{cases}$$

and

$$\begin{cases} y_{(l^2, 0)}^{n+1}(z) = y_{(l_1^2, l_2^2, \dots, l_n^2 - 1)}^{n+1}(z), \\ y_{(l^2, s^2)}^{n+1}(z) = y_{(l^2, s^2 - 1)}^{n+1}(z) + \left(b \prod_{v=1}^n r_{l_v^2}^v(z) \sum_{i=1}^{N_1(n)} q_i(z) \right) r_{s^2}^{n+1}(z), \end{cases}$$

and set

$$J_{(l, s)}^{n+1}(z) = [x_{(l^1, s^1 - 1)}^{n+1}(z), x_{(l^1, s^1)}^{n+1}(z)[x[y_{(l^2, s^2 - 1)}^{n+1}(z), y_{(l^2, s^2)}^{n+1}(z)],$$

where $s = s^1 \times s^2$, for each $s = \overline{1, N(n+1)}$ ($s^1 = \overline{1, N_1(n+1)}$, $s^2 = \overline{1, N_2(n+1)}$).

Then $J_{(l, s)}^{n+1}(z)$ is nonempty if and only if $z \in \bigcap_{v=1}^n W_{l_v}^v \cap W_s^{n+1}$, and, in particular,

$z \in \bigcap_{v=1}^n W_{l_v}^v$ implies that (3.8) holds nontrivially. We observe, that, in this

case, $\text{diam } J_l^n(z) = ab \prod_{v=1}^n p_{l_v}^v(z) > 0$. However, whatever $z \in \mathcal{K}$, we have by construction that

$$(3.9) \quad J = \bigcup \{J_l^n(z) : l = (l_1, l_2, \dots, l_n), 1 \leq l_v \leq N(v), 1 \leq v \leq n\}.$$

We define, for every $n \geq 1$, the required mapping g^n of $\mathcal{K}$ into $\mathcal{L}(D; \mathbb{R}^n)$. In view of (3.9) we can do this simply by prescribing, for every $z \in \mathcal{K}$, the restriction of $g^n(z)$ to each of the intervals $J_l^n(z)$. For every $z \in \mathcal{K}$, we define

$$(3.10) \quad g^1(z)/J = \sum_{v=1}^{N(1)} \chi[J_v^1(z)] v_v^1,$$

where χ is the characteristic function and set, for every $n \geq 1$, and every $2n$ -vector index $l = (l_1, l_2, \dots, l_n)$, $l = l^1 \times l^2$, $l^1 = (l_1^1, l_2^1, \dots, l_n^1)$, $l^2 = (l_1^2, l_2^2, \dots, l_n^2)$, $1 \leq l_v \leq N(v)$, $1 \leq l_v^1 \leq N_1(v)$, $1 \leq l_v^2 \leq N_2(v)$, $1 \leq v \leq n$,

$$(3.11) \quad g^{n+1}(z) \big|_{J_l^n(z)} = \sum_{v=1}^{N(n+1)} \chi[J_{(l, s)}^{n+1}(z)] v_{(l, s)}^{n+1}.$$

This uniquely defines, for every $n \geq 1$, $g^n(z)$ as a piecewise constant mapping in J , and hence we can extend $g^n(z)$ to $D = \bar{J}$ by setting

$$(3.12) \quad \begin{cases} g^n(z)(a, y) = \lim_{x \rightarrow a-} g^n(z)(x, y), \\ g^n(z)(x, b) = \lim_{y \rightarrow b-} g^n(z)(x, y). \end{cases}$$

Obviously, for every $n \geq 1$, g^n is a mapping of $\mathcal{X}$ into $\mathcal{L}^1(D; \mathbb{R}^n)$. This construction implies, similarly with [5, Proposition 2] that each g^n is continuous in $\mathcal{X}$. Thus, only the inequalities (3.1) and (3.2) remain to be verified.

Let $z \in \mathcal{X}$ be given and fix $(x, y) \in J$. Then, for every $n \geq 1$, there exists one and only one $2n$ -vector index $l = (l_1, l_2, \dots, l_n)$, $l = l^1 \times l^2$, such that $(x, y) \in J_l^n(z)$.

This implies that, in particular $z \in \bigcap_{v=1}^n W_{l_v}^v$ and consequently, by (3.11),

$$d(g^n(z)(x, y), G(z)(x, y)) = d(v_l^n(x, y), G(z)(x, y)) \leq d(v_l^n(x, y), G(z_l^n)(x, y)) + \\ + H(G(z_l^n)(x, y)) G(z)(x, y) < 2^{-n-1} + 2^{-n-3} < 2^{-n}.$$

Moreover, if $(x, y) \in J_l^n$ then $(x, y) \in J_{(l, s)}^{n+1}$ for one and only one index s with $1 \leq s \leq N(n+1)$, so that also $z \in \bigcap_{v=1}^n W_{l_v}^v \cap W_s^{n+1}$. Hence, we deduce from (3.4) and (3.6) that

$$\|g^{n+1}(z)(x, y) - g^n(z)(x, y)\| = \|v_{(l, s)}^{n+1}(x, y) - v_l^n(x, y)\| < 2^{-n-1}.$$

Thus, the inequalities (3.1) (3.2) hold at every $(x, y) \in D$. Obviously, by (3.12) and continuity, they remain valid at $x=a$, $y=b$. This completes the proof.

4. Multivalued Equations with Partial Derivatives. Let us consider the multivalued equation

$$(4.1) \quad \frac{\partial^2 z}{\partial x \partial y} \in F(x, y, z), \quad (x, y) \in D, \quad z \in B,$$

where $F: D \times B \rightarrow \text{comp } X$.

The Cauchy problem associated with (4.1) is defined in [4] and consists in finding of absolutely continuous function [2, §565–§568] $z \in C^*(D; \mathbb{R}^n)$, which satisfies (4.1) a.e. $(x, y) \in D$ and (2.7). As corollary of theorem of selection we state the following existence result.

Theorem. Let be satisfied the hypotheses (H_0) , (H_1) , (H_2) where (H_2) $F: D \times B \rightarrow \text{comp } X$ is a continuous multifunction.

Then, there exists a regulated mapping $\lambda: D \rightarrow \mathbb{R}^n$ such that the mapping $\bar{z}: D \rightarrow \mathbb{R}^n$, given by

$$\begin{aligned}
 (4.2) \quad \bar{z}(x, y) &= R(x) - \int_y^{\varphi(x)} Q(\psi(v)) dv - \int_y^{\varphi(x)} dv \int_{\psi(v)}^x \lambda(u, v) du = \\
 &= \alpha(x, y) - \iint_{T(x, y)} \lambda(u, v) dudv,
 \end{aligned}$$

$(x, y) \in D$ is a solution of the Cauchy problem (4.1) + (2.7).

P r o o f. Let be $\lambda(x, y) = g(z)(x, y)$, $z \in \mathcal{K}$, $\lambda: D \rightarrow \mathbb{R}^n$, where g exists by the theorem of selection. From (H_1) , the function $\bar{z}$ defined by (4.2) is absolutely continuous in the Carathéodory sense [2, § 565 – § 568], $\bar{z} \in C^*(D; \mathbb{R}^n)$. By (4.2) it results $\partial^2 \bar{z}(x, y) / \partial x \partial y = \lambda(x, y) = g(\bar{z})(x, y) \in G(\bar{z})(x, y) = F(x, y, \bar{z}(x, y))$ a.e. $(x, y) \in D$ and $\bar{z}(x, \varphi(x)) = R(x)$, $\partial \bar{z}(x, \varphi(x)) / \partial x = P(x)$, $0 \leq x \leq a$. Hence, $\bar{z}$ is a solution of the Cauchy problem (4.1) + (2.7). The proof is similar for $y > \varphi(x)$.

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REFERENCES

1. Antosiewicz H. A., Cellina A., *Continuous Selections and Differential Relations*. J. Diff. Eq., **19**, 386–398 (1975).
2. Carathéodory C., *Vorlesungen über Reelle Funktionen*. Chelsea Publishing Company, 1948.
3. Rus I. A., *Principii și aplicații ale teoriei punctului fix*. Ed. Dacia, Cluj-Napoca, 1979.
4. Teodoru G., *Le problème de Cauchy pour une équation hyperbolique multivoque*. Bul. Inst. Polit. Iași, Supliment-Cercetări matematice, 1985, pp. 141–146.
5. Teodoru G., *Absolutely Continuous Solutions of Cauchy Problem for a Multivalued Hyperbolic Equation*, Bul. Inst. Polit. Iași, **XXXV (XXXIX)**, 1–2, s. I, 47–54 (1989).

SELECȚII CONTINUE PENTRU MULTIFUNCȚII ȘI PROBLEMA LUI CAUCHY PENTRU ECUAȚII MULTIVOCE

(Rezumat)

Se demonstrează o teoremă de existență a unei selecții continue pentru o multifuncție F definită pe o mulțime compactă din $\mathbb{R}^{n+2}$, F continuă, cu valori mulțimi nevide, compacte, nu necesar convexe. Drept corolar al acestei teoreme de selecție se demonstrează existența unei soluții pentru problema lui Cauchy asociată ecuației $\partial^2 z / \partial x \partial y \in F(x, y, z)$.

ÉLABORATION DES PROGRAMMES DIDACTIQUES POUR LA GÉNÉRATION DES SURFACES DANS L'ESPACE

PAR

C. POPA et MIRCEA DAN BUCEVSCI

1. Introduction. Un des problèmes fondamentaux caractéristique pour l'activité créatrice des ingénieurs c'est la transformation de l'espace tridimensionnel (S_3) dans l'espace bidimensionnel et vice-versa. Ayant en vue les difficultés posées par la visualisation de l'espace tridimensionnel pour les débutants, on propose des programmes didactiques pour la génération des surfaces dans l'espace, en utilisant les moyens actuels de calcul informatisé. La surface étant construite, elle peut être étudiée sous différents aspects, en réalisant des projections sous divers angles, en pratiquant des sections à différentes échelles, ce qui facilite à comprendre la corrélation entre l'image plane (projection) et le corps tridimensionnel. On réussit en fait l'assimilation du mécanisme qui assure la correspondance entre les deux espaces, avec des conséquences bénéfiques pour l'activité technique de projection et d'exécution. On familiarise aussi les futurs spécialistes avec l'utilisation du calculateur, en stimulant l'aspect créatif de leurs activités impliquées dans le mécanisme qui relie l'expression mathématique à l'aspect géométrique.

2. Analyse. Le principe de réalisation des programmes pour la génération des surfaces dans l'espace est le suivant : un corps peut être exprimé par voie analytique à l'aide d'une équation

$$(1) \quad Z = F(x, y).$$

Sa géométrie dans l'espace des valeurs admissibles pour x et y conduit à la famille de valeurs pour z .

Le contour du corps dans l'espace peut être obtenu à l'aide des courbes de niveau, qui sont les projections de l'intersection du corps par le plan $z = \text{const.}$ Puis, en partant des courbes de niveau, on peut représenter le contour dans l'espace du corps sectionné, en utilisant par exemple une représentation axonométrique izométrique.

3. Principe de la génération des surfaces dans l'espace. a) On propose différentes expressions mathématiques en deux variables de n'importe quel degré.

b) On calcule les valeurs admissibles de z (la troisième dimension du corps) en adoptant un certain interval de variation des coordonnées planes (x, y).

c) On cherche les valeurs minime et maxime de la troisième dimension, corrélées avec les domaines de x et y .

d) On partage le domaine min-max. de z dans un nombre entier de parties, par exemple comprises entre 5 ... 10; chacune représente le plan de section du corps avec un plan parallèle au plan horizontal, fixé par la hauteur z .

e) On représente le graphique de l'intersection du plan $z = \text{const.}$ avec le corps tridimensionnel.

f) On construit le contour du corps en utilisant les principes de la représentation axonométrique izométrique.

La sélectivité optique du corps visualisé dans la représentation axonométrique est strictement liée à l'échelle utilisée pour transformer les données numériques en unités graphiques.

On peut généraliser complètement le concept en utilisant des fonctions monofactorielles dans lesquelles on échange les opérateurs intérieurs, respectivement les opérateurs qui relient les deux fonctions. Par exemple

$$(2) \quad \begin{aligned} f_1 &= a_0 + a_1x + a_2x^2 + \dots, \\ f_2 &= b_0 + b_1x + b_2x^2 + \dots \end{aligned}$$

Les opérateurs qui interviennent sont l'addition et la division pour chacune des deux fonctions. À leur place on peut considérer l'opérateur division, l'opérateur puissance, l'opérateur logarithme etc.

À leur tour les deux fonctions peuvent être aussi associées à l'aide de différents opérateurs (addition, soustraction, multiplication, division, radicaux etc.) c'est-à-dire

$$(3) \quad f_1 + f_2; \quad f_1 = f_2; \quad f_1/f_2; \quad f_1f_2 \dots$$

Par généralisation, la troisième dimension z peut être exprimée par des combinaisons d'opérateurs au niveau de chaque fonction, de même qu'entre fonctions.

3.1. La méthodologie didactique. On peut nominaliser les phases de cette méthodologie comme suit :

1. Choix de l'opérateur au niveau d'une fonction.
2. Choix de l'opérateur entre fonctions.
3. Établissement du domaine x, y dans lequel existe la fonction z , sur les principes connus de l'analyse mathématique.
4. Attribution des valeurs des coefficients des fonctions.
5. Représentation graphique par des courbes de niveau de la fonction z , en utilisant par exemple un programme factoriel du deuxième ordre.
6. Obtention de l'image axonométrique du corps.
7. Modification de l'échelle pour améliorer la perception visuelle.

Dans la Fig. 1 on présente le schéma logique général pour générer les surfaces dans l'espace, en partant des expressions mathématiques librement choisies.

Pour exemplifier, on considère un polynôme du second degré, qui du point de vue géométrique, représente une section conique :

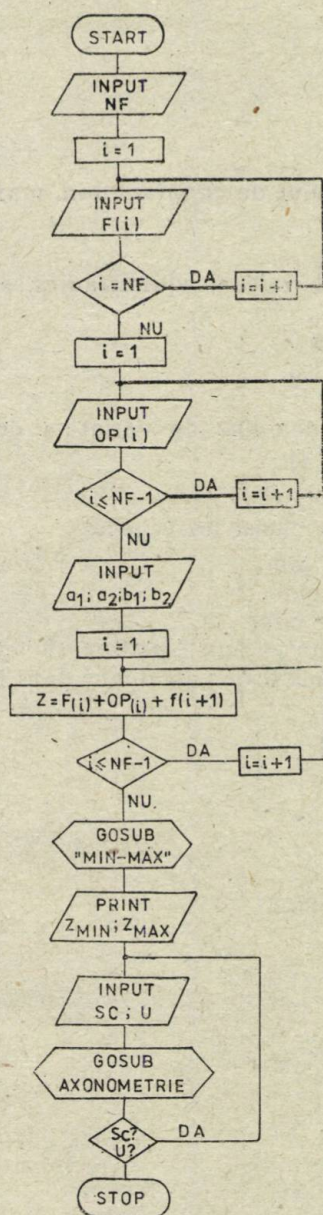
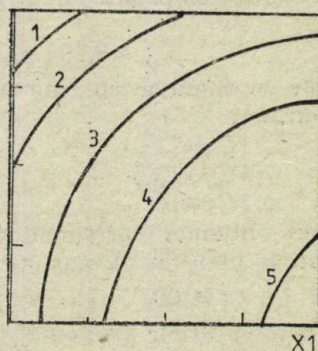

 $1 \rightarrow Y_m = 8.79436$
 $X1_{min} = 20$
 $X2_{min} = 70$
 $Y_m = 21.93442$
 $X1_{max} = 70$
 $X2_{max} = 40$
 $X2 \rightarrow [8.794 * 21.934]$
 $X1 \in [20--70]$
 $X2 \in [40--70]$


Fig. 2

 $1C1 = 11$
 $1C2 = 13.5$
 $1C3 = 16$
 $1C4 = 18.5$
 $1C5 = 21.5$

AXONOMETRIE

F1

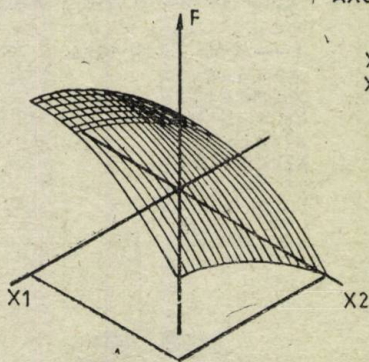
 $X1 \{20 - 70\}$
 $X2 \{40 - 70\}$
 $mF1 = 8.794$
 $MF1 = 21.934$


Fig. 3

Fig. 1 — Schéma logic simplifié pour la génération des surfaces dans l'espace: NF — nombre des fonctions; OP — opérateurs; a_1, a_2, b_1, b_2 — valeurs extrémales pour x et y ; Z — fonction créée; „MIN-MAX“ — sousroutine pour déterminer les extrêmes de la fonction z ; Sc — échelle de représentation; U — angle de rotation autour de l'axe z ; „AXONOMETRIE“ — sousroutine pour la représentation axonométrique.

$$\begin{aligned}
 & -7,7494 \\
 & +0,3572 x \\
 & +0,7432 y \\
 & -0,00194 x^2 \\
 & -0,000787 x, y \\
 & -0,008315 y^2.
 \end{aligned}
 \tag{4}$$

Cette fonction est empruntée de [2]. Le domaine de variation min.-max. est

$$z_{\min} = 8,79; \quad z_{\max} = 21,89. \tag{5}$$

On a divisé ce domaine en cinq portions de valeurs équidistantes, en résultant les courbes :

$$\begin{aligned}
 1C_1 &= 11 & 1C_4 &= 16,5 \\
 1C_2 &= 13,5 & 1C_5 &= 21,5. \\
 1C_3 &= 16
 \end{aligned}
 \tag{6}$$

Les valeurs obtenues correspondent à un domaine de variation des coordonnées dans le plan (x, y) , par exemple de [2]

$$x \in [20, 70] (^{\circ}\text{C}); \quad y \in [40, 70] (\%). \tag{7}$$

Pour la représentation axonométrique on a choisi les échelles

$$\begin{aligned}
 l_u \text{ numérique} & \quad x = 2,6 \text{ mm}, \\
 l_u \text{ numérique} & \quad y = 4,3 \text{ mm}, \\
 l_u \text{ numérique} & \quad z = 32,5 \text{ mm}.
 \end{aligned}
 \tag{8}$$

L'aspect des courbes de niveau correspondantes aux équation (1) est présenté dans la Fig. 2, pendant que l'image axonométrique est donné dans la

$$\begin{aligned}
 1 \text{ --- } & \rightarrow Y_m = -16,5794 \\
 X1 \text{ min} &= 0 \\
 X2 \text{ min} &= 100 \\
 Y_M &= 21,5096 \\
 X1 \text{ max} &= 75 \\
 X2 \text{ max} &= 50
 \end{aligned}$$

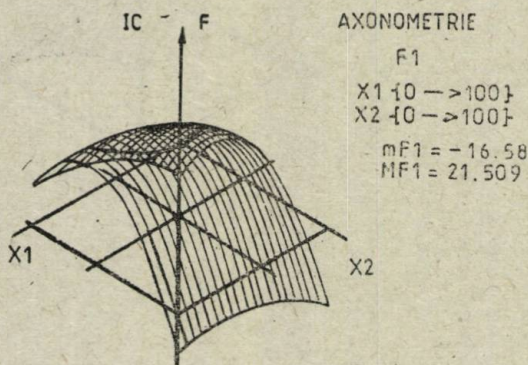


Fig. 4

Fig. 3. Pour le domaine initial de variation des variables x_1 et x_2 on présent l'image axonométrique de la surface à une autre échelle (Fig. 4).

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BIBLIOGRAPHIE

1. Gluck A., *Metode matematice în industria chimică*. Ed. Tehn., Buc., 1971.
2. Popa C., Chiriță Gh., Bucevschi M. D., *Studiul contracției pieilor în procesul uscării*. Industria Ușoară, 9 (1985).

CONCEPEREA UNOR PROGRAME DIDACTICE DE GENERARE A SUPRAFETELOR
ÎN SPAȚIU

(Rezumat)

Se prezintă principiile de generare a unor suprafețe în spațiu, plecând de la expresiile matematice ce caracterizează corpurile. Exemplele tratate s-au bazat pe date experimentale.

A PROBLEM OF SMALL THERMOELASTIC DEFORMATION SUPERPOSED ON FINITE THERMOELASTIC DEFORMATION

BY

ELISABETA RUSU and VICTOR HATMAN

The problem of small thermoelastic deformation in a body that is under initial stress and initial constant temperature was examined in [1]...[3]. The equations of small thermoelastic deformation superposed on a general thermomechanical deformation are derived in [4], and for isotropic solid, in [5].

In this paper we consider an application of the theory of small thermoelastic deformations superposed on a large deformation at nonconstant temperature [5].

We consider a body, denoted by B_0 in its undeformed state at T_0 temperature, of incompressible isotropic elastic material, which deforms, after a large thermoelastic deformation with the displacement vector $\mathbf{v}$ and temperature $T (\neq \text{const})$ into a state B which is in equilibrium. We suppose that the material is incompressible with Helmholtz free energy function σ and functions C_s :

$$(1) \quad \sigma = \sigma(T, I_1, I_2), \quad C_s = C_s(I_1, I_2, I_4, I_5, I_6), \quad s = 1, \dots, 6,$$

where C_s are polynomials in invariants with coefficients which do not depend on T [7].

Let θ^i be a general curvilinear coordinate system. Base vector and metric tensors in undeformed state B_0 and deformed state B are denoted respectively by: $\mathbf{g}_i, \mathbf{g}^i, g_{ij}, g^{ij}, \mathbf{G}_i, \mathbf{G}^i, G_{ij}, G^{ij}$. The moving coordinates $(\theta^1, \theta^2, \theta^3)$ are chosen to coincide with cylindrical polar coordinate (r, φ, z) in the deformed body B . Then the point (r, φ, z) was initially at (ρ, θ, Z) .

We consider the thermomechanical (controllable) state [7]

$$(2) \quad r = \frac{\rho}{\sqrt{\lambda}}; \quad \varphi = \theta; \quad z = \lambda Z,$$

coupled with the temperature field

$$(3) \quad T = T_0 + T_1 \varphi; \quad \lambda, T_1 = \text{const.}, \quad \lambda > 0.$$

We take the deformed configuration of the body, to be that bounded by the planes: $z = \pm h$, $\varphi = 0$, $\varphi = \varphi_1$, and right circular cylinders $r = r_1$ and

$r=r_2$. We consider the state (2), (3) as a deformation of the body which in the reference state was bounded by the planes $Z=\pm h/\lambda$, $\theta=0$, $\theta=\varphi_1$ and the cylinders $R_1=\sqrt{\lambda}r_1$ and $R_2=\sqrt{\lambda}r_2$. In the deformed state, the plane $\varphi=0$ is at temperature T_0 while $\varphi=\varphi_1$ is at temperature $T_0+T_1\varphi_1$; $\varphi_1\in(0, 2\pi)$.

The basic equations of nonlinear thermoelasticity theory are [6]:

i) *The equilibrium equations and energy equation respectively*

$$(4) \quad \tau^{ij}{}_{|j}=0, \quad Q^i{}_{|i}=0.$$

ii) *The constitutive equations*

$$(5) \quad \begin{aligned} \tau^{ij} &= s^{ij} = \Phi g^{ij} + \Psi B^{ij} + p G^{ij}, \\ -Q^i &= -q^i = (C_1 \delta_j^i + C_2 \gamma_j^i + C_3 \gamma_m^i \gamma_j^m) T^{|j} + {}_0 \varepsilon^{ijk} T|_r T|_s \gamma_t^r (C_4 \delta_j^s \delta_k^t + \\ &\quad + C_5 \delta_j^s \gamma_k^t + C_6 \gamma_j^s \gamma_k^t). \end{aligned}$$

iii) *The geometrical relations*

$$(6) \quad \gamma_{ij} = \frac{1}{2}(G_{ij} - g_{ij}); \quad \gamma_j^i = g^{ir} \gamma_{rj}.$$

To the system of field equations we adjoin the boundary conditions

$$(7) \quad \mathbf{t} = t^{ij} n_i \mathbf{G}_j, \quad h = Q^i n_i, \quad (\mathbf{n} = n_i \mathbf{G}^i) \quad \mathbf{Q} = Q^i \mathbf{G}_i.$$

In (1), (4), (5) we have also [6]

$$(8) \quad \begin{aligned} I_1 &= g^{ij} G_{ij}, \quad I_2 = g_{ij} G^{ij}, \quad I_3 = \frac{G}{g} = 1, \quad I_4 = T|{}^i T|_i, \quad I_5 = T|{}^i T|_j \gamma_i^j; \\ I_6 &= T|{}^i T|_j \gamma_k^j \gamma_i^k, \quad T|_r = \frac{\partial T}{\partial \theta^r}; \quad T|{}^j = g^{jr} T|_r; \end{aligned}$$

$$\Phi = 2 \frac{\partial \sigma}{\partial I_1}, \quad \Psi = 2 \frac{\partial \sigma}{\partial I_2}, \quad p = \text{a scalar function of coordinates};$$

$$B^{ij} = (g^{ij} g^{rs} - g^{ir} g^{js}) G_{rs}, \quad {}_0 \varepsilon^{ijk} = \frac{e_{ijk}}{\sqrt{g}}.$$

From (2), (3) and (8) we find

$$(9) \quad \begin{aligned} g_{11} &= \lambda, \quad g_{22} = \lambda r^2, \quad g_{33} = \frac{1}{\lambda^2}, \quad g_{ij} = 0 \quad (i \neq j); \\ g^{11} &= \frac{1}{\lambda}, \quad g^{22} = \frac{1}{\lambda r^2}, \quad g^{33} = \lambda^2, \quad g^{ij} = 0 \quad (i \neq j); \\ G_{11} &= G^{11} = G_{33} = G^{33} = 1, \quad G_{22} = r^2, \quad G^{22} = \frac{1}{r^2}, \quad G_{ij} = G^{ij} = 0 \quad (i \neq j); \\ \gamma_{11} &= \frac{1}{2}(1 - \lambda), \quad \gamma_{22} = \frac{1}{2}(1 - \lambda)r^2, \quad \gamma_{33} = \frac{1}{2}\left(1 - \frac{1}{\lambda^2}\right), \quad \gamma_{ij} = 0 \quad (i \neq j); \\ \gamma_1^1 &= \frac{1}{2\lambda}(1 - \lambda), \quad \gamma_2^2 = \frac{1}{2\lambda}(1 - \lambda), \quad \gamma_3^3 = \frac{\lambda^2}{2}\left(1 - \frac{1}{\lambda^2}\right), \quad \gamma_j^i = 0 \quad (i \neq j); \end{aligned}$$

$$\begin{aligned}
 B^{11} &= \lambda + \frac{1}{\lambda^2}, \quad B^{22} = \frac{1+\lambda^3}{\lambda^2 r^2}, \quad B^{33} = 2\lambda, \quad B^{ij} = 0 \quad (i \neq j); \\
 I_1 &= \lambda^2 + \frac{2}{\lambda}, \quad I_2 = 2\lambda + \frac{1}{\lambda^2}, \quad I_3 = 1; \\
 (10) \quad I_4 &= \frac{T_1^2}{\lambda r^2}, \quad I_5 = \frac{T_1^2}{2\lambda^2 r^2} (1-\lambda), \quad I_6 = \frac{T_1^2}{4\lambda^3 r^2} (1-\lambda)^2; \\
 \Gamma_{22}^1 &= -r, \quad \Gamma_{12}^2 = \Gamma_{21}^2 = \frac{1}{r}.
 \end{aligned}$$

From (5) we get

$$\begin{aligned}
 \tau^{11} &= s^{11} = \frac{1}{\lambda} \Phi + \frac{1+\lambda^3}{\lambda^2} \Psi + p, \quad \tau^{22} = \frac{s^{11}}{r^2}, \quad \tau^{33} = \lambda^2 \Phi + 2\lambda \Psi + p; \\
 \tau^{ij} &= 0 \quad (i \neq j); \quad Q^1 = Q^3 = 0,
 \end{aligned}$$

$$Q^2 = - \left[C_1 + C_2 \frac{1}{2\lambda} (1-\lambda) + C_3 \frac{1}{4\lambda^2} (1-\lambda)^2 \right] \frac{T_1}{r^2}$$

and

$$-Q^i T_{||i} \geq 0 \quad \text{becomes} \quad \frac{T_1}{r^2} \left[C_1 + C_2 \frac{1}{2\lambda} (1-\lambda) + C_3 \frac{1}{4\lambda^2} (1-\lambda)^2 \right] \geq 0.$$

The Eqs. (4) are

$$(11) \quad \frac{\partial s^{11}}{\partial r} + \frac{1}{r} (s^{11} - r^2 s^{22}) = 0, \quad \frac{\partial s^{22}}{\partial \theta} = 0, \quad \frac{\partial s^{33}}{\partial z} = 0, \quad \frac{\partial Q^2}{\partial \theta} = 0.$$

We see that σ and T are function of θ only. The Eq. (11)₁ is satisfied if $p = p(\theta)$. We get

$$(12) \quad p = p^0 - \frac{1}{\lambda} \Phi - \frac{1+\lambda^3}{\lambda^2} \Psi,$$

where p^0 is a constant. If there is no traction on curved surfaces, then $p^0 = 0$, and the components of the stress tensor and the function p in the state B are given by

$$(13) \quad \begin{cases} s^{11} = 0, & s^{22} = 0, & s^{33} = \Phi \left(\lambda^2 - \frac{1}{\lambda} \right) + \Psi \left(2\lambda - \frac{1+\lambda^3}{\lambda^2} \right); \\ s^{ij} = 0 & (i \neq j); & p = -\frac{1}{\lambda} \Phi - \frac{1+\lambda^3}{\lambda^2} \Psi. \end{cases}$$

The resultant normal force acting on $z = h$ is

$$N_3 = \int_A s^{33} dA = \iint \tau^{33} \sqrt{G} G^{33} d\theta^1 d\theta^2 = \frac{r_2^2 - r_1^2}{2} \int_0^{\varphi_1} \tau^{33} d\varphi,$$

and all surfaces are free of tractions, except the $z = \pm h$; the heat flow is circular.

The state B is completely known.

The body is subsequently subject to a small thermoelastic deformation. We suppose that the displacement vector is $\mathbf{v} + \varepsilon \mathbf{w}$ and temperature is $T + \varepsilon T'$. The corresponding state is B' . We seek the equilibrium configuration B' .

Now we consider a small thermoelastic deformation from the state B to B' with

$$(14) \quad w^1 = w_1 = u(r), \quad w^2 = w_2 = 0, \quad w^3 = w_3 = az, \quad T' = T'(\varphi).$$

The basic equations for equilibrium state B' are:

i) *The equilibrium equations* [6]

$$(15) \quad [s'^{ij} + s'^{ir}w^j]_{|r}]_i = 0.$$

ii) *The energy equation* [6]

$$(16) \quad [Q'^i + Q^i G^{mn} \gamma'_{mn}]_i = 0.$$

iii) *The constitutive equations*: the components of the stress tensor $\tau'^{ij} + \varepsilon \tau'^{ij}$, where [8]

$$(17) \quad \tau'^{ij} = s'^{ij} = g^{ij} \Phi' + B^{ij} \Psi' + B'^{ij} \Psi + G'^{ij} p + G^{ij} p';$$

the components of heat flux vector $Q^i + \varepsilon Q'^i$ where Q'^i are

$$(18) \quad \begin{aligned} -Q'^i = T|_r [C_2 \gamma'^i_j + C_3 (\gamma'_m \gamma'^m_j + \gamma'^m_j \gamma'_m) + (C_{1p} \delta^i_j + C_{2p} \gamma^i_j + C_{3p} \gamma'^i_m \gamma'^m_j) I'_p] + \\ + T'|_r [C_1 \delta^i_j + C_2 \gamma^i_j + C_3 \gamma^i_m \gamma^m_j] + \frac{e_{ijk}}{\sqrt{g}} \{ T|_r T|_s \gamma^r_t [C_5 \delta^s_j \gamma'^t_k + C_6 (\gamma^s_j \gamma'^t_k + \\ + \gamma'^s_j \gamma^t_k) + (C_{4p} \delta^s_j \delta^t_k + C_{5p} \delta^s_j \gamma^t_k + C_{6p} \gamma^s_j \gamma^t_k) I'_p] + (C_4 \delta^s_j \delta^t_k + C_5 \delta^s_j \gamma^t_k + \\ + C_6 \gamma^s_j \gamma^t_k) [\gamma^r_t (T'|_r T|_s + T|_r T'|_s) + T|_r T|_s \gamma'^r_t] \}. \end{aligned}$$

iv) *The geometrical relations*

$$(19) \quad \gamma^i_j + \varepsilon \gamma'^i_j = g^{ir} (\gamma_{rj} + \varepsilon \gamma'_{rj}); \quad \gamma'^i_{ij} = \frac{1}{2} G'_{ij}.$$

To the system of Eqs. (15)–(19) we adjoin the boundary conditions. We have also

$$(20) \quad I'_3 = 0$$

and [8]

$$G'_{ij} = w_i|_j + w_j|_i; \quad G' = G G^{ij} G'_{ij}, \quad G'^{ij} = -G^{ir} G^{js} G'_{rs};$$

$$I'_1 = g^{rs} G'_{rs}, \quad I'_2 = g_{rs} (G'^{rs} I_3 + G^{rs} I'_3), \quad I'_3 = I_3 G^{ij} G'_{ij};$$

$$A = 2 \frac{\partial^2 \sigma}{\partial I_1^2}, \quad B = 2 \frac{\partial^2 \sigma}{\partial I_2^2}, \quad F = 2 \frac{\partial^2 \sigma}{\partial I_1 \partial I_2}, \quad L = 2 \frac{\partial^2 \sigma}{\partial I_1 \partial T}, \quad M = 2 \frac{\partial^2 \sigma}{\partial I_2 \partial T};$$

$$(21) \quad \Phi' = A I'_1 + F I'_2 + L T'; \quad \Psi' = F I'_1 + B I'_2 + M T',$$

p' = a scalar function and [5]

$$I'_4 = T|_i T'|^i + T|^i T'|_i,$$

$$\begin{aligned}
 (22) \quad I'_5 &= T|^i T|_j \gamma'^j_i + \gamma^i_j (T|^i T'|_j + T'|^i T|_j), \\
 I'_6 &= T|^i T|_j (\gamma'^j_k \gamma^k_i + \gamma^j_k \gamma'^k_i) + \gamma^j_k \gamma^k_i (T'|^i T|_j + T|^i T'|_j), \\
 C_{sp} &= \frac{\partial C_s}{\partial I_p} \quad (s=1, \dots, 6; \quad p=1, 2, 4, 5, 6).
 \end{aligned}$$

From (14) and (21) we get

$$\begin{aligned}
 (23) \quad G'_{11} &= 2 \frac{du}{dr}, \quad G'_{22} = 2ru, \quad G'_{33} = 2a, \quad G'^{11} = -2 \frac{du}{dr}, \quad G'^{22} = -2 \frac{u}{r^3}, \\
 G'^{33} &= -2a, \quad G'_{ij} = G'^{ij} = 0 \quad (i \neq j); \quad \gamma'^1_1 = \frac{1}{\lambda} \frac{du}{dr}, \quad \gamma'^2_2 = \frac{u}{\lambda r}, \quad \gamma'^3_3 = a\lambda^2; \\
 I'_1 &= 2 \left(\frac{1}{\lambda} \frac{du}{dr} + \frac{u}{\lambda r} + a\lambda^2 \right), \quad I'_2 = 2 \left(\lambda + \frac{1}{\lambda^2} \right) \left(\frac{du}{dr} + \frac{u}{r} \right) + 4a\lambda, \\
 B'^{11} &= \frac{2}{\lambda} \left(\frac{u}{\lambda r} + a\lambda^2 \right), \quad B'^{22} = \frac{2}{\lambda r^2} \left(\frac{1}{\lambda} \frac{du}{dr} + a\lambda^2 \right), \\
 B'^{33} &= 2\lambda \left(\frac{du}{dr} + \frac{u}{r} \right), \quad B'^{ij} = 0 \quad (i \neq j).
 \end{aligned}$$

From (17) and (23) we find

$$\begin{aligned}
 (24) \quad s'^{11} &= \frac{1}{\lambda} \Phi' + \frac{1+\lambda^3}{\lambda^2} \Psi' + \frac{2}{\lambda} \left(\frac{u}{\lambda r} + a\lambda^2 \right) \Psi - 2 \frac{du}{dr} p + p', \\
 s'^{22} &= \frac{1}{\lambda r^2} \Phi' + \frac{1+\lambda^3}{\lambda^2 r^2} \Psi' + \frac{2}{\lambda r^2} \left(\frac{1}{\lambda} \frac{du}{dr} + a\lambda^2 \right) \Psi - 2 \frac{u}{r^3} p + \frac{p'}{r^2}, \\
 s'^{33} &= \lambda^2 \Phi' + 2\lambda \Psi' + 2\lambda \left(\frac{du}{dr} + \frac{u}{r} \right) \Psi - 2ap + p', \\
 s'^{ij} &= 0 \quad (i \neq j),
 \end{aligned}$$

and from (18) we get

$$\begin{aligned}
 (25) \quad -Q'^2 &= T|^2 \{ C_3 \gamma'^2_2 + C_3 \gamma^2_2 \gamma'^2_2 \} + [C_{1p} + C_{2p} \gamma^2_2 + C_{3p} (\gamma^2_2)^2] I'_p \} + \\
 &+ T'|^2 [C_1 + C_2 \gamma^2_2 + C_4 \gamma^2_2 \gamma'^2_2], \quad Q'^1 = Q'^3 = 0.
 \end{aligned}$$

From (22) we find

$$\begin{aligned}
 (26) \quad I'_4 &= T|^2 T'|^2 + T|^2 T'|_2, \quad I'_5 = T|^2 T|_2 \gamma'^2_2 + \gamma^2_2 (T|^2 T'|_2 + T'|^2 T|_2), \\
 I'_6 &= T|^2 T|_2 (\gamma'^2_2 \gamma^2_2 + (\gamma^2_2)^2) + (\gamma^2_2)^2 (T'|^2 T|_2 + T|^2 T'|_2).
 \end{aligned}$$

The Eqs. (15) (16) and (20) are respectively

$$\frac{\partial s'^{11}}{\partial r} + \frac{1}{r} (s'^{11} - r^2 s'^{22}) = 0, \quad \frac{\partial s'^{22}}{\partial \varphi} = 0, \quad \frac{\partial s'^{33}}{\partial z} = 0, \quad \frac{\partial Q'^2}{\partial \varphi} = 0;$$

$$(27) \quad \frac{du}{dr} + \frac{u}{r} + a = 0,$$

From (27)₃ we get

$$(28) \quad u = \frac{A_0}{r} + \frac{ar}{2},$$

where A_0 is constant.

By (27)₄, using (25) and (28) we obtain

$$(29) \quad T' = K_1 \varphi + K_2,$$

where K_1 and K_2 are constants.

By (27)₁ and (27)₃, using (28), (29) and (24) we get

$$\frac{\partial p'}{\partial r} = 0, \quad \frac{\partial p'}{\partial z} = 0.$$

From (27)₂ we find

$$(30) \quad p' = K_3 - \frac{1}{\lambda} \Phi' - \frac{1+\lambda^3}{\lambda^2} \Psi' + a \Psi \left(\frac{1}{\lambda^2} - 2\lambda \right) - ap.$$

The components of the stress tensor in the state B_0 are

$$(31) \quad \begin{aligned} s'^{11} &= K_3 + \frac{2A_0\Psi}{\lambda^2 r^2} + \frac{2pA_0}{r^2}, \quad s'^{22} = \frac{1}{r^2} \left(K_3 - \frac{2A_0\Psi}{\lambda^2 r} - \frac{2A_0 p}{r^2} \right), \\ s'^{33} &+ \frac{\lambda^3 - 1}{\lambda} \Phi' + \frac{\lambda^3 - 1}{\lambda^2} \Psi' + a \Psi \frac{1 - 4\lambda}{\lambda^2} - 3ap + K_3. \end{aligned}$$

If $A_0 = 0$, $K_3 = 0$ then there is no traction on curved surfaces and on the planes $\varphi = 0$ and $\varphi = \varphi_1$. The additional flux $h' = 0$ is on the curved surfaces. The resultant normal force acting on the ends may be written as

$$N_3 + \varepsilon N'_3 = \iint (s^{33} + s'^{33}) \sqrt{G + \varepsilon G'^{33}} r dr d\varphi,$$

and if N'_3 is given we can find a .

The additional flux $h' = 0$ on the ends; we can find K_1 and K_2 if $T' = 0$ on $\varphi = 0$, $T' = T_2$ on $\varphi = \varphi_1$.

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REFERENCES

1. England A. H., Green A. E., *Steady-state Thermoelasticity for Initially Stressed Bodies*. Phil. Trans. R. Soc., A, 253, 517–542 (1961).
2. Flavin J. N., Green A. E., *Plane Thermoelastic Wave in an Initially Stressed Medium*. J. Mech. Phys. Solids, 9, 179–190 (1961).

3. Green A. E., *Thermoelastic Stresses in Initially Stressed Bodies*. Proc. R. Soc. A, **266**, 1—19 (1962).
4. Ieșan D., *Incremental Equations in Thermoelasticity*. J. of Thermal Stress, **3**, 1, 41—56, (1980).
5. Rusu E., *Small Thermoelastic Deformation Superposed on Finite Thermoelastic Deformation in a Cylindrical Tube*. Bul. Inst. Polit. Iași, **XXV (XXIX)**, 1—2, s. IV, 19—26 (1979).
6. Green A. E., Adkins J. E., *The Elastic Deformations*. Oxford, 1960.
7. Petroski H. J., Carlson E., *Some Exact Solutions to the Equations of Nonlinear Thermoelasticity*. J. Appl. Mech., **37**, 1 151—1 154 (1970).
8. Green A. E., Zerna W., *Theoretical Elasticity*. Oxford, 1954.

O PROBLEMĂ DE DEFORMAȚII TERMOELASTICE MICI SUPRAPUSE PE O DEFORMAȚIE TERMOELASTICĂ FINITĂ

(Rezumat)

Se consideră un corp omogen, izotrop, supus la o deformare termoelastice mică suprapusă peste o deformare termoelastice finită, care este o stare controlabilă.

A GENERALIZED THEORY OF THERMOELASTIC COSSERAT SOLIDS

BY

I. NISTOR

1. Introduction. A theory of thermoelasticity which will allow for so-called "second sound" effects has received much attention in recent years. For non-polar media this problem has been studied in various works (see e.g. [1], [2]). The fundamentals of the linear theory of a generalized thermoelasticity of micropolar continuum has been developed by Boschi and Ieșan [3].

The present paper is concerned with a generalized thermoelastic theory of Cosserats' medium.

The paper is organized as follows. In the next section the basic formulas and equations are given. The following section deals with thermodynamics of heat-conducting Cosserat solids. For non-linear thermoelasticity we obtain explicit expressions for the stress tensor, the couple stress tensor, the heat conduction vector and the entropy in terms of two scalar functions. The theory is linearized and the results established by Boschi and Ieșan [3]. The last section treats the thermodynamics of non-heat conducting Cosserat solids.

In the theory of thermoelasticity of Green and Lindsay an acceleration wave is defined as a singular surface across which the discontinuities of lowest order are the second derivatives of displacement and temperature (see e.g. [4]).

We [5] have investigated acceleration waves in the linear anisotropic theory of thermoelasticity of Green and Lindsay [1]. We have defined an acceleration wave as a propagating singular surface on which the deformation gradient, velocity, temperature and entropy are continuous and on which a discontinuity in acceleration and gradient temperature occurs and we have shown that every acceleration wave in a definite conductor of heat is homothermal ($[\dot{\theta}] = 0$). In this paper we extend the definition of acceleration wave by requiring that the functions $\dot{\phi}_i$, $\phi_{j,i}$ (ϕ_i are the components of microrotation vector) be continuous while their first derivatives $\ddot{\phi}_i$, $\dot{\phi}_{j,i}$, $\phi_{i,jk}$ may exhibit jump discontinuities across the singular surface. In Sect. 4 we prove (in linear theory) that every acceleration wave in a non-conductor of heat is homocalloric ($[\dot{\eta}] = 0$).

2. Basic Equations. The motion of a typical body point X of Cosserat medium is given [6] by

$$(2.1) \quad y_i = y_i(x_j, t), \quad d_{i,j} = d_{i,j}(x_k, t),$$

where x_i and y_j are the coordinate of the place occupied by the body point X in the reference configuration B_r and the current configuration B_t , respectively; d_{ij} are the components of the director d_i .

Let the components of the directors at reference configuration be denoted by D_{ij} . We have $d_{ij} = R_{jk}(x_p, t)D_{ik}(x_q)$, where $R_{jk}(x_q, t)$ are the components of a proper orthogonal tensor which measures the rotation of a point in the Cosserat medium.

Without loss of generality we may assume that the initial values of the director field are unit vectors along the coordinate axes

$$(2.2) \quad D_{ij} = \delta_{ij},$$

where δ_{ij} is the Kronecker delta.

The components v_i of the angular velocity vector are given by

$$(2.3) \quad v_i = \frac{1}{2} e_{ijk} R_{jp} \dot{R}_{kp},$$

where a superposed dot indicates material time differentiation and e_{ijk} is the alternating Ricci tensor.

We have [6] the relations

$$(2.4) \quad \dot{d}_{hk} = e_{kpq} d_{hq} v_p, \quad d_{jk,h} = e_{kpq} (d_{jq,h} v_p + d_{jq} v_{p,h}),$$

where $_{,j}$ denotes partial differentiation with respect to x_j .

One form of the standard balance equations for thermoelastic Cosserat continuum is

$$(2.5) \quad s_{ij,i} + \rho_0 F_j = \rho_0 \dot{v}_j, \quad \mu_{ji,i} + e_{ijk} y_{j,h} s_{hk} + \rho_0 G_i = \rho_0 \dot{\sigma}_i,$$

$$(2.6) \quad \rho_0 \dot{U} = e_{ijk} y_{i,h} s_{hk} v_j + \mu_{hj} v_{j,k} + s_{ij} v_{j,i} - q_{i,i} + \rho_0 r.$$

In these equations s_{ij} , μ_{ij} and q_i are the components of the stress tensor, couple stress tensor and heat flux vector measured per unit area in the reference configuration; F_i and G_i are the specific externally applied body force and couple, respectively; r is the externally supplied specific heat; ρ_0 is the reference mass density; v_j is the velocity field; σ_i is the microspin field and U is the specific internal energy.

Let us consider the entropy production inequality proposed by Green and Laws [2]

$$(2.7) \quad \frac{d}{dt} \int_P \rho_0 \eta dv - \int_P \frac{\rho_0 r}{\Phi} dv + \int_{\partial P} \frac{q_{(n)}}{\Phi} da \geq 0,$$

for every material volume P , where η is specific entropy, n_i is the outward unit normal to the boundary surface ∂P of P and Φ is a scalar function. We add restrictions $\Phi > 0$ and that in equilibrium Φ reduces to θ (θ is the absolute temperature). From (2.7) we have

$$(2.8) \quad \rho \Phi \dot{\eta} + q_{i,i} - \rho_0 r - q_i \frac{\Phi_{,i}}{\Phi} \geq 0.$$

With the help of (2.6) and the definition

$$(2.9) \quad \Psi = U - \eta \Phi,$$

the inequality (2.8) becomes

$$(2.10) \quad -\rho_0(\dot{\Psi} + \eta\dot{\Phi}) + {}_{ij}(v_{j,i} + e_{jh}y_{h,i}v_k) + \mu_{hj}v_{j,h} - q_i \frac{\Phi_{,i}}{\Phi} \geq 0.$$

The constitutive assumptions are that

$$(2.11) \quad \Psi, \Phi, s_{ij}, \mu_{ij}, q_i, \eta,$$

are function of $y_{i,j}, d_{hk}, d_{rs,i}, \theta, \dot{\theta}, \theta_{,i}$.

From (2.10) and (2.11) we have

$$(2.12) \quad -\rho_0 \left\{ \left(\frac{\partial \Psi}{\partial y_{i,j}} + \eta \frac{\partial \Phi}{\partial y_{i,j}} \right) v_{i,j} + \left(\frac{\partial \Psi}{\partial d_{hk}} + \eta \frac{\partial \Phi}{\partial d_{hk}} \right) e_{kpq} d_{hq} v_p + \right. \\ \left. + \left(\frac{\partial \Psi}{\partial d_{jk,h}} + \eta \frac{\partial \Phi}{\partial d_{jk,h}} \right) (d_{jq,h} v_p + d_{jq} v_{p,h}) e_{kpq} + \left(\frac{\partial \Psi}{\partial \theta} + \eta \frac{\partial \Phi}{\partial \theta} \right) \dot{\theta} + \right. \\ \left. + \left(\frac{\partial \Psi}{\partial \dot{\theta}} + \eta \frac{\partial \Phi}{\partial \dot{\theta}} \right) \ddot{\theta} + \left(\frac{\partial \Psi}{\partial \theta_{,i}} + \eta \frac{\partial \Phi}{\partial \theta_{,i}} \right) \dot{\theta}_{,i} \right\} + s_{ij}(v_{j,i} + e_{jh}y_{h,i}v_k) + \\ + \mu_{hj}v_{j,h} - \frac{q_i}{\Phi} \left(\frac{\partial \Phi}{\partial y_{h,k}} y_{h,ki} + \frac{\partial \Phi}{\partial d_{jk}} d_{jk,i} + \frac{\partial \Phi}{\partial d_{jk,h}} d_{jk,hi} + \right. \\ \left. + \frac{\partial \Phi}{\partial \theta} \theta_{,i} + \frac{\partial \Phi}{\partial \dot{\theta}} \dot{\theta}_{,i} + \frac{\partial \Phi}{\partial \theta_{,j}} \theta_{,ji} \right) \geq 0.$$

3. Thermodynamics of Heat Conducting Cosserat Solid. We consider heat-conducting Cosserat solids which are characterized by constitutive assumption that $q_i \neq 0$ ($i=1, 2, 3$). As in [1], from (2.12) we get

$$(3.1) \quad s_{ij} = \rho_0 \frac{\partial \Psi}{\partial y_{j,i}}, \quad \mu_{ij} = \rho_0 e_{jqk} d_{pq} \frac{\partial \Psi}{\partial d_{pk,i}},$$

$$(3.2) \quad \frac{\partial \Psi}{\partial \dot{\theta}} + \eta \frac{\partial \Phi}{\partial \dot{\theta}} = 0, \quad \rho_0 \frac{\partial \Psi}{\partial \theta_{,i}} + \frac{q_i}{\Phi} \frac{\partial \Phi}{\partial \theta_{,i}} = 0,$$

$$(3.3) \quad e_{kpq} \left[s_{ik} y_{q,i} + \rho_0 \left(\frac{\partial \Psi}{\partial d_{hk}} + \eta \frac{\partial \Phi}{\partial d_{hk}} \right) d_{hq} + \rho_0 \frac{\partial \Psi}{\partial d_{jk,h}} d_{jq,h} \right] = 0,$$

$$(3.4) \quad \frac{\partial \Phi}{\partial y_{h,k}} = 0, \quad \frac{\partial \Phi}{\partial d_{jk,h}} = 0, \quad \frac{\partial \Phi}{\partial \theta_{,i}} = 0,$$

$$(3.5) \quad -\rho_0 \left(\frac{\partial \Psi}{\partial \theta} + \eta \frac{\partial \Phi}{\partial \theta} \right) \dot{\theta} - \frac{q_i}{\Phi} \left(\frac{\partial \Phi}{\partial d_{jk}} d_{jk,i} + \frac{\partial \Phi}{\partial \theta} \theta_{,i} \right) \geq 0.$$

From (3.4) it follows that

$$(3.6) \quad \Phi = \Phi(d_{jk}, \theta, \dot{\theta}).$$

By the principle of material indifference, the function Φ satisfies

$$(3.7) \quad \Phi(Q_{ij}d_{kj}, \theta, \dot{\theta}) = \Phi(d_{ki}, \theta, \dot{\theta}),$$

identically in d_{ij} for all orthogonal tensor $Q = (Q_{ij})$.

Choosing $Q_{ij} = R_{ji}$ (R_{ij} the components of microrotation tensor) we obtain

$$(3.8) \quad Q_{ij}d_{kj} = R_{ji}d_{kj} = D_{ki}.$$

From (3.6)–(3.8) and (2.2), Φ can be represented by

$$(3.9) \quad \Phi = \Phi(\theta, \dot{\theta}).$$

Using (3.9) the relations (3.3) and (3.5) become

$$(3.10) \quad e_{kpq} \left(\frac{\partial \Psi}{\partial y_{k,i}} y_{q,i} + \frac{\partial \Psi}{\partial d_{hk}} d_{hq} + \frac{\partial \Psi}{\partial d_{jk,h}} d_{jq,h} \right) = 0,$$

$$(3.11) \quad -\rho_0 \left(\frac{\partial \Psi}{\partial \theta} + \gamma \frac{\partial \Phi}{\partial \theta} \right) \dot{\theta} - \frac{q_i}{\Phi} \frac{\partial \Phi}{\partial \theta} \theta_{,i} \geq 0.$$

For any function $f(., \dot{\theta}, \theta_{,i})$ we put $f|_E = f(., 0, 0)$. We have already impose the condition $\Phi|_E = 0$ hence

$$(3.12) \quad \left. \frac{\partial \Phi}{\partial \theta} \right|_E = 1.$$

As in [2], from (3.11) we get

$$(3.13) \quad q_i|_E = 0, \quad \rho|_E = - \left. \frac{\partial \Psi}{\partial \theta} \right|_E.$$

The Eq. (3.10) does not represent an independent set of constitutive relations, but coincides with the requirement that Ψ is an objective function. It is easily verified that (3.10) is equivalent to

$$(3.14) \quad \sigma = \sigma(c_{ij}, \gamma_{ij}, \theta, \dot{\theta}, \theta_{,i}),$$

where

$$(3.15) \quad c_{ij} = y_{k,i} R_{kj}, \quad \gamma_{ij} = \frac{1}{2} e_{jmn} R_{km,i} R_{ln}, \quad \sigma = \rho_0 \Psi.$$

For the deformation it is convenient to introduce the strain tensor

$$(3.16) \quad e_{ij} = c_{ij} - \delta_{ij}.$$

Using the relation (2.3) and the identities

$$e_{ijk} R_{qi} R_{pj} R_{tk} = |R_{mn}| e_{qp}, \quad e_{ijk} e_{mjk} = 2\delta_{im},$$

we obtain

$$(3.17) \quad \dot{e}_{ij} = R_{kj} v_{k,i} + e_{kpq} R_{qi} y_{k,i} v_p, \quad \dot{\gamma}_{ij} = R_{pj} v_{p,i}.$$

Let us introduce the tensors

$$(3.18) \quad t_{ik} = s_{ip} R_{pk} = \frac{\partial \sigma}{\partial e_{ik}}, \quad m_{ik} = \mu_{ij} R_{jk} = \frac{\partial \sigma}{\partial \gamma_{ik}}.$$

Taking into account (3.14), (3.17) and (3.18) Eqs. (2.5) and (2.6) become

$$(3.19) \quad (R_{ij} t_{kj})_{,k} + \rho_0 F_i = \rho_0 \ddot{u}_i, \\ (R_{ij} m_{kj})_{,k} + e_{ijk} y_{j,p} R_{kp} t_{pq} + \rho_0 G_i = \rho_0 \dot{\sigma}_i,$$

$$(3.20) \quad \rho_0 r - q_{i,i} - \rho_0 \eta \Phi - \left(\frac{\partial \sigma}{\partial \theta} + \rho_0 \eta \frac{\partial \Phi}{\partial \theta} \right) \dot{\theta} - \frac{\partial \sigma}{\partial \theta_{,i}} \dot{\theta}_{,i} = 0.$$

The functions t_{ij} , m_{ij} , q_j and η are all expressed in terms of two scalar functions σ and Φ by Eqs. (3.2) and (3.18).

In the linear theory of Cosserat media [7] we have

$$(3.21) \quad R_{ij} = \delta_{ij} + e_{jip} \varphi_p, \quad e_{ij} = u_{j,i} + e_{jip} \varphi_p, \quad \gamma_{ij} = \varphi_{j,i} = \varkappa_{ij}.$$

We add the condition $\Phi(\theta, 0) = \theta_0 + \theta$, where θ_0 is constant.

In the linear theory of Cosserat media, assuming that the initial body is free from stress and couple stress and has zero heat flux, from (3.9), (3.12), (3.13) and (3.14) it follows that σ and Φ may be represented in the form

$$(3.22) \quad \sigma = \sigma_0 - (\theta + \alpha \dot{\theta}) (a - a_{ij} e_{ij} - c_{ij} \varkappa_{ij}) - \frac{1}{2} d \dot{\theta}^2 - e \theta \dot{\theta} - \frac{1}{2} f \dot{\theta}^2 + \\ + \alpha b_i \dot{\theta}_{,i} + \frac{1}{2} \alpha k_{ij} \theta_{,i} \theta_{,j} + \frac{1}{2} A_{ijkl} e_{ij} e_{kl} + B_{ijkh} e_{ij} \varkappa_{kh} + \frac{1}{2} C_{ijkh} \varkappa_{ij} \varkappa_{kh},$$

$$(2.23) \quad \Phi = \theta_0 + \theta + \alpha \dot{\theta} + \beta \theta \dot{\theta} + \frac{1}{2} \gamma \dot{\theta}^2,$$

where $e - \alpha \beta = d \alpha$.

The above relations are deduced by Boschi and Ieșan [3].

4. Thermodynamics of Non-heat Conducting Cosserat Solids. We consider non-conductors of heat, namely, those Cosserat solids for which the heat flux function q_i vanishes identically. In this case the Eq. (2.12) takes the form

$$(4.1) \quad -\rho_0 \left\{ \left(\frac{\partial \Psi}{\partial y_{i,j}} + \eta \frac{\partial \Phi}{\partial y_{i,j}} \right) v_{i,j} + \left(\frac{\partial \Psi}{\partial d_{hk}} + \eta \frac{\partial \Phi}{\partial d_{hk}} \right) e_{kpq} d_{hq} v_p + \right. \\ \left. + \left(\frac{\partial \Psi}{\partial d_{jkh}} + \eta \frac{\partial \Phi}{\partial d_{jkh}} \right) (d_{jq,h} v_p + d_{jq} v_{p,h}) e_{kpq} + \left(\frac{\partial \Psi}{\partial \theta} + \eta \frac{\partial \Phi}{\partial \theta} \right) \dot{\theta} + \right. \\ \left. + \left(\frac{\partial \Psi}{\partial \dot{\theta}} + \eta \frac{\partial \Phi}{\partial \dot{\theta}} \right) \ddot{\theta} + \left(\frac{\partial \Psi}{\partial \theta_{,i}} + \eta \frac{\partial \Phi}{\partial \theta_{,i}} \right) \dot{\theta}_{,i} \right\} + \\ + s_{ij} (v_{j,i} + e_{jkh} y_{h,i} v_k) + \mu_{hj} v_{j,h} \geq 0.$$

We require that (4.1) hold identically for all temperature, displacement and microrotation fields, it being assumed that the moment, moment of momentum and energy equations are balanced by a suitable choice of externally applied body force F_i , body couple G_i and specific heat supply r . From (4.1) we then obtain the identities

$$(4.2) \quad \frac{\partial \Psi}{\partial \theta} + \eta \frac{\partial \Phi}{\partial \theta} = 0,$$

$$(4.3) \quad s_{ji} = \rho_0 \left(\frac{\partial \Psi}{\partial y_{i,j}} + \eta \frac{\partial \Phi}{\partial y_{i,j}} \right), \quad \mu_{ij} = \rho_0 e_{j,pq} d_{kp} \left(\frac{\partial \Psi}{\partial d_{qk,i}} + \eta \frac{\partial \Phi}{\partial d_{qk,i}} \right),$$

$$(4.4) \quad \frac{\partial \Psi}{\partial \theta_{,i}} + \eta \frac{\partial \Phi}{\partial \theta_{,i}} = 0,$$

$$(4.5) \quad e_{kpq} \left[s_{ik} y_{q,i} + \rho_0 d_{hq} \left(\frac{\partial \Psi}{\partial d_{hk}} + \eta \frac{\partial \Phi}{\partial d_{hk}} \right) + \rho_0 \left(\frac{\partial \Psi}{\partial d_{jk,h}} + \eta \frac{\partial \Phi}{\partial d_{jk,h}} \right) d_{jq,h} \right] = 0,$$

$$(4.6) \quad -\rho_0 \left(\frac{\partial \Psi}{\partial \theta} + \eta \frac{\partial \Phi}{\partial \theta} \right) \dot{\theta} \geq 0.$$

By an argument to the one leading from (3.7) to (3.9) we obtain that

$$(4.7) \quad \Phi = \Phi(c_{ij}, \gamma_{hk}, \theta, \dot{\theta}, \theta_{,i}),$$

where c_{ij} and γ_{hk} are given by the relations (3.15)_{1,2}. Substitution of (4.3)₁ into (4.5) then yields the equation

$$(4.8) \quad e_{kpq} \left[\left(\frac{\partial \Psi}{\partial y_{k,i}} + \eta \frac{\partial \Phi}{\partial y_{k,i}} \right) y_{q,i} + d_{hq} \left(\frac{\partial \Psi}{\partial d_{hk}} + \eta \frac{\partial \Phi}{\partial d_{hk}} \right) + \left(\frac{\partial \Psi}{\partial d_{jk,h}} + \eta \frac{\partial \Phi}{\partial d_{jk,h}} \right) d_{jq,h} \right] = 0.$$

Let us introduce the tensors

$$(4.9) \quad \begin{aligned} t_{ik} &= s_{ip} R_{pk} = \frac{\partial \sigma}{\partial e_{ik}} + \rho_0 \eta \frac{\partial \Phi}{\partial e_{ik}}, \\ m_{ik} &= \mu_{ij} R_{jk} = \frac{\partial \sigma}{\partial \gamma_{ik}} + \rho_0 \eta \frac{\partial \Phi}{\partial \gamma_{ik}}. \end{aligned}$$

Using the relations (3.14), (3.15), (4.2), (4.4), (4.7) and (4.8) the energy equation (2.6) may be represented in the form

$$(4.10) \quad \rho_0 r - \rho_0 \dot{\eta} \Phi - \left(\frac{\partial \sigma}{\partial \theta} + \rho_0 \eta \frac{\partial \Phi}{\partial \theta} \right) \dot{\theta} = 0.$$

The functions σ and Φ are specified by (3.14) and (4.7), respectively. The stress tensor, couple stress tensor and entropy η are expressed in terms of functions σ and Φ by Eqs. (4.9) and (4.2), respectively.

Returning to the inequality (4.6) we may show that

$$(4.11) \quad \left(\frac{\partial \Psi}{\partial \theta} + \eta \frac{\partial \Phi}{\partial \theta} \right) \Big|_{\dot{\theta}=0} = 0.$$

The functions σ and Φ given by (3.14) and (4.7) satisfy identically Eq. (4.8). Not any arbitrary functions are admissible, because all constitutive equations for σ and Φ must satisfy the conditions (4.4), (4.6) and (4.11).

In linear theory, assuming that the initial body is free from stress and couple stress, from (3.14), (4.4), (4.8) and (4.11) it follows that σ and Φ may be represented in the form

$$(4.12) \quad \sigma = \sigma_0 - a(\theta + \alpha\dot{\theta}) - \frac{1}{2}d\dot{\theta}^2 - c\dot{\theta}\ddot{\theta} - \frac{1}{2}f\ddot{\theta}^2 + aa_i\dot{\theta}\ddot{\theta}_i + ab_i\ddot{\theta}\ddot{\theta}_i + \frac{1}{2}ak_{ij}\dot{\theta}_i\ddot{\theta}_j +$$

$$+ aa_{ij}^0 e_{ij} + a_{ij}e_{ij}\dot{\theta} + a_{ij}^1 e_{ij}\dot{\theta} + aa_{ijk}e_{ij}\dot{\theta}_k + ak_{ij}^0 \chi_{ij} + k_{ij}^1 \chi_{ij}\dot{\theta} + k_{ij}^2 \chi_{ij}\ddot{\theta} +$$

$$+ aa_{ijk}^1 \chi_{ij}\dot{\theta}_k + \frac{1}{2}A_{ijkh}e_{ij}e_{kh} + B_{ijkh}e_{ij}\chi_{kh} + \frac{1}{2}C_{ijkh}\chi_{ij}\chi_{kh},$$

$$(4.13) \quad \Phi = \theta_0 + \theta + \alpha\dot{\theta} + \beta\ddot{\theta} + \frac{1}{2}\gamma\ddot{\theta}^2 - a_i\dot{\theta}\ddot{\theta}_i - b_i\ddot{\theta}\ddot{\theta}_i - \frac{1}{2}k_{ij}\dot{\theta}_i\ddot{\theta}_j - a_{ij}^0 e_{ij} +$$

$$+ b_{ij}e_{ij}\dot{\theta} + b_{ij}^1 e_{ij}\dot{\theta} - a_{ijk}e_{ij}\dot{\theta}_k - a_{ijk}^1 \chi_{ij}\dot{\theta}_k - k_{ij}^0 \chi_{ij}\dot{\theta} + h_{ij}\chi_{ij}\dot{\theta} + h_{ij}^1 \chi_{ij}\ddot{\theta} +$$

$$+ \frac{1}{2}a_{ijkh}e_{ij}e_{kh} + b_{ijkh}e_{ij}\chi_{kh} + \frac{1}{2}c_{ijkh}\chi_{ij}\chi_{kh},$$

where

$$(4.14) \quad e - a\beta = d\alpha, \quad (a_{ij} + ab_{ij})\alpha = a_{ij}^1 + ab_{ij}^1,$$

$$h\alpha = f - a\gamma, \quad \alpha(k_{ij}^1 + ah_{ij}) = k_{ij}^2 + ah_{ij}^1.$$

In deriving the relation (4.12) and (4.13) we make use of the condition

$$(4.15) \quad h - d\alpha \neq 0.$$

Substitution of (4.12) and (4.13) into (4.2), (4.6) and (4.9) gives

$$(4.16) \quad \rho_0 \eta = a + d\dot{\theta} + h\ddot{\theta} - (a_{ij} + ab_{ij})e_{ij} - (k_{ij}^1 + ah_{ij})\chi_{ij},$$

$$(4.17) \quad t_{ij} = (a_{ij} + ab_{ij} - da_{ij}^0)\dot{\theta} + [(a_{ij} + ab_{ij})\alpha - ha_{ij}^0]\ddot{\theta} + [A_{ijpq} + aa_{ijpq} +$$

$$+ a_{ij}^0(a_{pq} + ab_{pq})]e_{pq} + [B_{ijpq} + ab_{ijpq} + a_{ij}^0(k_{pq} + ah_{pq})]\chi_{pq},$$

$$(4.18) \quad m_{ij} = (k_{ij}^1 + ah_{ij} - dk_{ij}^0)\dot{\theta} + [(k_{ij}^1 + ah_{ij})\alpha - hk_{ij}^0]\ddot{\theta} + [B_{ijpq} + ab_{ijpq} +$$

$$+ k_{ij}^0(a_{pq} + ab_{pq})]e_{pq} + [C_{ijpq} + ac_{ijpq} + k_{ij}^0(k_{pq}^1 + ah_{pq})]\chi_{pq},$$

$$(4.19) \quad (\alpha d - h)\ddot{\theta}^2 \geq 0.$$

From (4.15) and (4.19) we have

$$(4.20) \quad \alpha d - h > 0.$$

In the linear theory Eq. (4.10) becomes

$$(4.21) \quad r - \dot{\eta}\theta_0 = 0.$$

Substitution (4.16) into (4.21) yields

$$(4.22) \quad h\ddot{\theta} + d\dot{\theta} - (a_{ij} + ab_{ij})e_{ij} - (k_{ij}^1 + ah_{ij})\chi_{ij} - \frac{\rho_0}{\theta_0}r = 0.$$

The Eq. (4.22) is capable of predicting a finite speed of heat propagation if and only if

$$(4.23) \quad h \neq 0.$$

The singular surfaces considered in the present work are acceleration waves, defined as propagating surfaces on which the quantities $u_{i,j}$, v_j , $\varphi_{j,i}$, $\dot{\varphi}_{j,i}$, $\dot{\theta}$ and η are continuous and some at least of their first partial derivatives discontinuous. When, in addition $[\dot{\theta}] = 0$, the acceleration wave is called homothermal; if, instead, $[\dot{\eta}] = 0$, the wave is called homocaloric ([8], p. 184).

We assume further that the body force F_i , couple body G_i and body heating r are continuous

$$[F_i]=0, \quad [G_i]=0, \quad [r]=0.$$

From the relations (4.16), (4.21) and (4.23) results that every acceleration wave in a non-conductor of heat is homothermal and homocaloric.

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REFERENCES

1. Green A. E., Lindsay K. A., *Thermoelasticity*. J. Elasticity, **2**, 1–7 (1972).
2. Green A. E., Laws N., *On the Entropy Production Inequality*. Arch. Rational Mech. Anal., **45**, 47–59 (1972).
3. Boschi E., Ieșan D., *A Generalized Theory of Linear Micropolar Thermoelasticity*. Meccanica, **VIII**, 154–157 (1973).
4. Green A. E., *A Note on Linear Thermoelasticity*. Mathematika, **19**, 69–75 (1972).
5. Nistor I., *Acceleration Waves in Linear Thermoelastic Materials*. Bul. Inst. Polit. Iași, **XXXV (XXXIX)**, 1–2, s. I, 61–67 (1989).
6. Nistor I., *Systematic Methods of Approximation in Nonlinear Theory of Cosserat Media (I)*. Bul. Inst. Polit. Iași, **XXVIII (XXXII)**, 3–4, s. I, 53–63 (1982).
7. Nistor I., *On Finite Deformation of Elastic Cosserat Continuum*. Bul. Inst. Polit. Iași, **XXII (XXVI)**, 1–2, s. IV, 20–24 (1976).
8. Truesdell C., *Rational Thermodynamics*. Second Edition, Springer Verlag, New York – Berlin–Heidelberg–Tokyo, 1984.

O TEORIE GENERALIZATĂ A SOLIDELOR COSSERAT TERMOELASTICE

(Rezumat)

Teoria termoelasticității stabilită de Green și Lindsay [1] cu ajutorul inegalității producerii entropiei propusă de Green și Laws [2] este generalizată pentru solide Cosserat termoelastice.

Teoria este liniarizată și sînt obținute rezultatele stabilite de Boschi și Ieșan [3]. Se demonstrează că orice undă de accelerație ce se propagă în medii neconducătoare de căldură este homotermică și homocalorică.

THE THEORY OF LINEAR THERMOELASTICITY OF PLATES THE VIBRATION IN THE TRAVERSE DIRECTION

BY

I. IBĂNESCU

0. Introduction. In this paper we consider the deformation of plate in the generalized theory of classical thermoelasticity presented by I. Müller [1] and A. E. Green and K. A. Lindsay [2].

In [3] we have establish the field of equations of vibration in the traverse direction. Now, a uniqueness theorem is stated.

1. Basic Equations. The field of equations in generalized thermoelasticity as presented in [2] is reproduced here as follows :

i) *The equations of motion*

$$(1.1) \quad \begin{aligned} t_{ik,t} &= \rho \ddot{u}_k - \rho F_k, & t_{ki} &= t_{ik}; \\ \rho r - q_{i,t} - \rho \theta_0 \dot{\eta} &= 0, & \text{in } B \times [0, \infty). \end{aligned}$$

ii) *The constitutive equations*

$$(1.2) \quad \begin{aligned} t_{ik} &= \lambda e_{rr} \delta_{ik} + 2\mu e_{ik} - \frac{E\beta}{3(1-2\nu)} (\theta + \alpha \dot{\theta}) \delta_{ik}, \\ \rho \eta &= a + d\theta + h' \dot{\theta} + \frac{E\beta}{3(1-2\nu)} e_{rr}, \\ q_i &= -k\theta_{,i}, & \text{in } B \times [0, \infty). \end{aligned}$$

iii) *The kinematic relations*

$$(1.3) \quad e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \text{in } B \times [0, \infty).$$

Here t_{ik} represents the components of the stress tensor; u_k — components of displacement vector; F_k — components of body forces; q_k — components of heat flux vector; η — specific entropy; θ — temperature field in body; $\lambda, \mu, \beta, a, d, h'$ — constants of material; B is the domain of the plate; δ_{ik} is Kronecker's symbol.

The symbol $_{,k}$ denotes the partial differentiation with respect to x_k ($k=1, 2, 3$) and the superposed dot denotes the differentiation with respect to time t . Latin indices have the range 1, 2, 3 and Greek indices take the value 1, 2, only.

The density ρ and the initial temperature θ_0 are given constants subjected to the conditions

$$(1.4) \quad \rho > 0, \quad \theta_0 > 0.$$

From the residual entropy inequality [2] we have the condition

$$(1.5) \quad (d\alpha - h')x^2 + ky_i y_i \geq 0,$$

for all arbitrary x and y_i and

$$(1.6) \quad d\alpha - h' > 0, \quad k > 0.$$

Let h be the thickness of plate. We choose the coordinate axis x_1 and x_2 on the middle surface of the plate, and x_3 perpendicular to it, forming an orthogonal system.

We write the Eqs. (1.1) in the form

$$(1.7) \quad t_{\alpha\beta,\alpha} + t_{3\beta,3} + \rho F_\beta = \rho \ddot{u}_\beta,$$

$$(1.8) \quad t_{\alpha 3,\alpha} + t_{33,3} + \rho F_3 = \rho \ddot{u}_3.$$

Integrating (1.8) over the thickness of plate, we obtain

$$(1.9) \quad A_{\alpha,\alpha} + p = \rho \frac{\partial^2 U}{\partial t^2},$$

where

$$(1.10) \quad A_\alpha = \int_{-h/2}^{h/2} t_{\alpha 3} dx_3, \quad [t_{33}]_{-h/2}^{h/2} + \rho \int_{-h/2}^{h/2} F_3 dx_3 = p, \quad U = \int_{-h/2}^{h/2} u_3 dx_3.$$

Multiplying (1.7) by x_3 and integrating with respect to x_3 from $x_3 = -h/2$ to $x_3 = h/2$, we have

$$(1.11) \quad M_{\alpha\beta,\alpha} - A_\beta + s_\beta = \rho \ddot{V}_\beta,$$

where

$$(1.12) \quad M_{\alpha\beta} = \int_{-h/2}^{h/2} t_{\alpha\beta} x_3 dx_3, \quad [t_{33} x_3]_{-h/2}^{h/2} + \rho \int_{-h/2}^{h/2} F_\beta x_3 dx_3 = s_\beta, \quad V_\beta = \int_{-h/2}^{h/2} x_3 u_\beta dx_3.$$

According to the theory of thin plates [4], we take

$$(1.13) \quad u_\alpha(x_1, x_2, t) = w(x_1, x_2, t), \quad u_\alpha = -x_3 w_{,\alpha},$$

and the constitutive equations have the form (see [3]):

$$(1.14) \quad t_{\alpha\beta} = \frac{E}{1-\nu^2} \left\{ [(1-\nu)w_{,\alpha\beta} + \nu w_{,\gamma\gamma} \delta_{\alpha\beta}] x_3 - \frac{\beta(1+\nu)}{3} (\theta + \alpha \dot{\theta}) \delta_{\alpha\beta} \right\},$$

$$(1.15) \quad \rho \eta = d^* \dot{\theta} + h^* \ddot{\theta} - \frac{E\beta}{3(1-\nu)} x_3 w_{,\gamma\gamma},$$

where

$$(1.16) \quad d^* = d + \frac{E\beta^2\alpha(1+\nu)}{9(1-2\nu)(1-\nu)}, \quad h^* = h' + \frac{E\beta^2\alpha(1+\nu)}{9(1-2\nu)(1-\nu)}.$$

Using the relation (1.10) (1.11) and (1.12) we have

$$(1.17) \quad U = \int_{-h/2}^{h/2} u_3 dx_3 = hw, \quad V_\beta = \int_{-h/2}^{h/2} x_3 u_\beta dx_3 = -\frac{h^3}{12} w_{,\beta}$$

and

$$(1.18) \quad M_{\alpha\beta,\alpha} - A_\beta + s_{\beta} = -\frac{h^3}{12} w_{,\beta}.$$

Substituting (1.18) into (1.9) we find that

$$(1.19) \quad M_{\alpha\beta,\alpha\beta} + p + s_{\beta,\beta} = h\rho\ddot{w} + \frac{h^3}{12} w_{,\beta\beta}.$$

The term $h^3/12$ in the last equation can be neglected compared with the other. Then Eq. (1.19) can be written

$$(1.20) \quad M_{\alpha\beta,\alpha\beta} + p + s_{\beta,\beta} = h\rho\ddot{w}.$$

The Eqs. (1.20) is similar to those used by B. A. Boley and J. H. Weiner in the classical theory.

Multiplying Eq. (1.15) by x_3 and integrating with respect to x_3 from $x_3 = -h/2$ to $x_3 = +h/2$ we have

$$(1.21) \quad \rho\tilde{\eta} = d^*\tau + h^*\dot{\tau} - \frac{E\beta}{3(1-\nu)} \nabla^2 w,$$

where

$$(1.22) \quad \tilde{\eta} = \frac{12}{h^3} \int_{-h/2}^{h/2} \eta x_3 dx_3, \quad \tau = \frac{12}{h^3} \int_{-h/2}^{h/2} \theta x_3 dx_3.$$

Now, we consider Eq. (1.1)₃. Multiplying it by x_3 and integrating over the thickness, we obtain

$$(1.23) \quad \rho\tilde{R} - \tilde{Q}_{\alpha,\alpha} - \frac{12}{h^3} [x_3 q_3]_{-h/2}^{h/2} + \frac{12}{h^3} k \left[\theta \left(x_1, x_2, \frac{h}{2}, t \right) - \theta \left(x_1, x_2, -\frac{h}{2}, t \right) \right] - \rho\theta_0\dot{\tilde{\eta}} = 0,$$

$$(1.24) \quad \tilde{R} = \frac{12}{h^3} \int_{-h/2}^{h/2} r x_3 dx_3, \quad \tilde{Q}_\alpha = \frac{12}{h^3} \int_{-h/2}^{h/2} q_\alpha x_3 dx_3.$$

Because the plate thickness is small, we shall take for τ the expression

$$(1.25) \quad h\tau(x_1, x_2, t) = \theta\left(x_1, x_2, \frac{h}{2}, t\right) - \theta\left(x_1, x_2, -\frac{h}{2}\right).$$

We note

$$(1.26) \quad q_3\left(x_1, x_2, \frac{h}{2}, t\right) = \bar{p}(x_1, x_2, t), \quad q_2\left(x_1, x_2, -\frac{h}{2}\right) = -\bar{q}(x_1, x_2, t).$$

The Eq. (1.23) can be written as

$$(1.27) \quad \rho\tilde{R} - \tilde{Q}_{\alpha,\alpha} - \frac{6}{h^2}(\bar{p} - \bar{q}) - \frac{12k}{h^2}\tau - \theta_0\dot{\eta} = 0.$$

From (1.12) and (1.14) we have

$$(1.28) \quad M_{\alpha,\beta} = -N\left[(1-\nu)w_{,\alpha\beta} + \nu\Delta w\delta_{\alpha\beta} + \frac{\beta(1+\nu)}{3}(\tau + \alpha\dot{\tau})\delta_{\alpha\beta}\right],$$

where

$$(1.29) \quad N = \frac{Eh^3}{12(1-\nu^2)}.$$

Multiplying (1.2)₃ by x_3 and integrating with respect to x_3 , we have

$$(1.30) \quad \tilde{Q}_\alpha = -k\tau_{,\alpha}.$$

The Eqs. (1.20), (1.28) represent the equation of motion and (1.21), (1.28), (1.30) represent the constitutive equations.

2. The Boundary Conditions. We denote by D the domain of the middle section and by Γ its boundary. Let us consider an elementary section having the normal $n = (n_1, n_2, 0)$. The bending moment acting on this section is given by

$$(2.1) \quad M_n = M_{\alpha\beta}n_\alpha n_\beta.$$

The twisting moment is defined as

$$(2.2) \quad M_{nt} = \varepsilon_{\rho\lambda}M_{\alpha\rho}n_\alpha n_\lambda, \quad \varepsilon_{11} = \varepsilon_{22} = 0, \quad \varepsilon_{12} = -\varepsilon_{21} = 1$$

and the shearing stress resultants is

$$(2.3) \quad A_n = A_\alpha n_\alpha.$$

We consider the boundary conditions

$$w = \hat{w}, \quad \frac{\partial w}{\partial n} = \hat{w}_1 \quad \text{on the } \Gamma_1;$$

$$(2.4) \quad M_{\alpha\beta}n_\alpha = \hat{M}_\beta, \quad A_n - \frac{\partial M_{nt}}{\partial s} = \hat{N} \quad \text{on the } \Gamma_2;$$

$$\tau = \hat{\tau} \quad \text{on the } \Gamma_3; \quad Q_\alpha n_\alpha = \hat{Q} \quad \text{on the } \Gamma_4,$$

where $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4$ denotes subsets of Γ so that

$$(2.5) \quad \Gamma_1 \cup \Gamma_2 = \Gamma, \quad \Gamma_3 \cup \Gamma_4 = \Gamma, \quad \Gamma_1 \cap \Gamma_2 = \Gamma_3 \cap \Gamma_4 = \emptyset.$$

3. A Uniqueness Theorem for the Vibration in the Traverse Direction.

Let w and τ be two functions satisfying the homogeneous Eqs. of motion (1.20), (1.27), constitutive relations (1.15), (1.28), boundary conditions (2.4) in which $\hat{w}, \hat{w}_1, \hat{M}_\beta, \hat{N}, \hat{\tau}, \hat{Q}$ are each zero and the initial conditions

$$(3.1) \quad w(x_1, x_2, 0) = w^0, \quad \dot{w}(x_1, x_2, 0) = \dot{w}^0, \quad \tau(x_1, x_2, 0) = \tau^0, \quad \dot{\tau}(x_1, x_2, 0) = \dot{\tau}^0.$$

We use the notation $f(l)$, $f(m)$ for the function evaluated at time $t=l$ and $t=m$. Let calculate the integral

$$(3.2) \quad I = \iint_D \left[M_{\alpha\beta}(m) \dot{w}_{,\alpha\beta}(l) - \frac{h^3}{12} \ddot{\tau}(l) [\tau(m) + \alpha \dot{\tau}(m)] \right] dA.$$

Using the divergence theorem we find

$$(3.3) \quad I = \int_\Gamma \left\{ M_{\alpha\beta}(m) \dot{w}_{,\alpha}(l) n_\beta - M_{\alpha\beta,\beta}(m) \dot{w}(l) n_\alpha + \frac{h^3}{12\theta_0} \ddot{Q}_\alpha(l) n_\alpha [\tau(m) + \alpha \dot{\tau}(m)] \right\} ds + \\ + \iint_D \left\{ \rho h \ddot{w}(m) \dot{w}(l) - \frac{h^3}{12\theta_0} Q_\alpha(l) [\tau_{,\alpha}(m) + \alpha \dot{\tau}_{,\alpha}(m)] \right\} dA.$$

Using the constitutive equations, the integral (3.2) take the following form :

$$(3.4) \quad I = - \iint_D N [(1-\nu) w_{,\alpha\beta}(m) \dot{w}_{,\alpha\beta}(l) + \nu \Delta w(m) \Delta \dot{w}(l)] dA - \\ - \frac{h^3}{12} \iint_D \{ [d^* \ddot{\tau}(l) + h^* \ddot{\tau}(l)] [\tau(m) + \alpha \dot{\tau}(m)] \} dA.$$

From (3.3) and (3.4) we see that

$$(3.5) \quad \int_\Gamma M_{\alpha\beta}(m) \dot{w}_{,\alpha}(l) n_\beta - M_{\alpha\beta,\beta}(m) \dot{w}(l) n_\alpha + \frac{h^3}{12\theta_0} Q_\alpha(l) n_\alpha [\tau(m) + \alpha \dot{\tau}(m)] ds = \\ = - \iint_D \left\{ \rho h \ddot{w}(m) \dot{w}(l) + \frac{h^3 k}{12\theta_0} \tau_{,\alpha}(l) [\tau_{,\alpha}(m) + \alpha \dot{\tau}_{,\alpha}(m)] \right\} dA - \\ - N \iint_D [(1-\nu) w_{,\alpha\beta}(m) \dot{w}_{,\alpha\beta}(l) + \nu \Delta w(m) \Delta \dot{w}(l)] dA - \\ - \frac{h^3}{12} \iint_D [d^* \ddot{\tau}(l) \tau(m) + d^* \alpha \dot{\tau}(l) \dot{\tau}(m) + h^* \ddot{\tau}(l) \tau(m) + h^* \alpha \dot{\tau}(m) \ddot{\tau}(l)] dA.$$

The left hand side of (3.5) vanishes. Now, we consider $l=m=z$ and integrating (3.5) with respect to z from $z=0$ to $z=t$, using (zero) initial conditions, we have

$$(3.6) \quad \iint_D \left\{ \rho h \dot{w}^2 + \frac{kh^3 \alpha}{12\theta_0} \tau_{,\sigma} \tau_{,\sigma} + \frac{h^3}{12} (d^* \tau^2 + h^* \alpha \dot{\tau}^2 + 2h^* \tau \dot{\tau}) + N[(1-\nu)w_{,\alpha\beta} w_{,\alpha\beta} + \nu(\Delta w)^2] \right\} dA + \frac{h^3}{6} \int_0^t \left\{ \iint_D \left[(d^* \alpha - h^*) \dot{\tau}^2 + \frac{k}{\theta_0} \tau_{,\sigma} \tau_{,\sigma} \right] dA \right\} dz = 0.$$

In (3.5) we put $l=z$, $m=2l-z$ and subtract from it the expression in which $l=2l-z$, $m=z$. If the resulting expression is integrated with respect to z from $z=0$ to $z=t$, it results

$$(3.7) \quad \iint_D \left\{ -\rho h \dot{w}^2 - \frac{h^3 k \alpha}{12\theta_0} \tau_{,\sigma} \tau_{,\sigma} + N[(1-\nu)w_{,\alpha\beta} w_{,\alpha\beta} + \nu(\Delta w)^2] + \frac{h^3}{12} (d^* \tau^2 + 2h^* \tau \dot{\tau} + h^* \alpha \dot{\tau}^2) \right\} dA = 0.$$

From (3.6) and (3.7) we see that

$$(3.8) \quad \iint_D \left(\rho h \dot{w}^2 + \frac{kh^3 \alpha}{12\theta_0} \tau_{,\sigma} \tau_{,\sigma} \right) dA + \frac{h^3}{12} \int_0^t \left\{ \iint_D \left[(d^* \alpha - h^*) \dot{\tau}^2 + \frac{k}{\theta_0} \tau_{,\sigma} \tau_{,\sigma} \right] dA \right\} dz = 0.$$

If we adopt the conditions $\alpha > 0$, $d^* \alpha - h^* > 0$, in view of (1.4), (1.5) and from (3.8) it follows that

$$(3.9) \quad \dot{w} = 0, \quad \tau_{,\alpha} = 0, \quad \dot{\tau} = 0$$

and in virtue of initial conditions we have

$$(3.10) \quad w = 0, \quad \tau = 0$$

and the uniqueness of mixed boundary solution is proved.

4. Another Proof for the Uniqueness Theorem. We will prove now the uniqueness theorem using the identity

$$(3.11) \quad \begin{aligned} & \int_{\Gamma} \left[M_{\alpha\beta} \dot{w}_{,\alpha} n_{\beta} + \frac{h^3}{12\theta_0} (\tau + \alpha \dot{\tau}) \tilde{Q}_{\alpha} n_{\alpha} - M_{\alpha\beta,\beta} n_{\alpha} \dot{w} \right] ds = \\ & = -\frac{1}{2} \frac{d}{dt} \iint_D \left\{ N[(1-\nu)w_{,\alpha\beta} w_{,\alpha\beta} + \nu(\Delta w)^2] + \frac{h^3}{12} \left[\alpha (d^* \alpha - h^*) \left(\frac{\tau}{\alpha} \right)^2 + \right. \right. \\ & \quad \left. \left. + \frac{h^*}{\alpha} (\tau + \alpha \dot{\tau})^2 \right] + \rho h \dot{w}^2 + \frac{h\tau^2}{\theta_0} + \frac{k\alpha}{12\theta_0} \tau_{,\gamma} \tau_{,\gamma} \right\} dA - \\ & \quad - \frac{kh^3}{12\theta_0} \iint_D \tau_{,\gamma} \tau_{,\gamma} dA - \frac{h^3}{12} \iint_D (d^* \alpha - h^*) \dot{\tau}^2 dA - \iint_D \frac{kh^3 \tau^2}{12\theta_0} dA. \end{aligned}$$

Using the notation

$$(3.12) \quad U = \iint_D \left\{ N[(1-\nu)w_{,\alpha\beta}w_{,\alpha\beta} + \nu(\Delta w)^2] + \frac{h^3}{12} \left[\alpha(d^*\alpha - h^*) \left(\frac{\tau}{\alpha} \right)^2 + \right. \right. \\ \left. \left. + \frac{h^*}{\alpha} (\tau + \alpha\dot{\tau})^2 \right] + \rho h \dot{w}^2 + \frac{h}{\theta_0} \tau^2 + \frac{k\alpha}{12\theta_0} \tau_{,\sigma} \tau_{,\sigma} \right\} dA$$

for the homogeneous initial and boundary conditions, the identity (3.11) becomes

$$(3.13) \quad \dot{U} = - \frac{kh^3}{6\theta_0} \iint_D \tau_{,\gamma} \tau_{,\gamma} dA - 2 \iint_D (d^*\alpha - h^*) \dot{\tau}^2 dA - 2 \iint_D \frac{h\tau^2}{\theta_0} dA.$$

Because $k > 0$, $\theta_0 > 0$, $d^*\alpha - h^* \geq 0$, from (3.13) it results that $\dot{U} \leq 0$ and therefore the function U decreases for $t \in [0, t_0]$. But $U(t)$ is a positive function and takes the value zero for zero initial conditions. It results that $U(t) = 0$.

Because $\alpha > 0$, $d^*\alpha - h^* \geq 0$, from (3.12) it results that $\dot{w} = 0$, $\dot{\tau} = 0$ and in virtue of zero initial condition we have $w = 0$, $\tau = 0$.

This proves the uniqueness of the mixed boundary solution.

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REFERENCES

1. Müller I., *The Coldness, a Universal Function in Thermoelastic Bodies*. Arch. Rat. Mech. Anal., **41**, 319–332 (1971).
2. Green A. E., Lindsay K. A., *Thermoelasticity*. J. Elasticity, **2**, 1–7 (1972).
3. Ibănescu I., *A Generalized Theory of Linear Thermoelasticity of Plates*. Bul. Inst. Polit. Iași, **XXVIII (XXXII)**, 1–2, s. I, 39–45 (1982).
4. Boley B. A., Weiner J. H., *Theory of Thermal Stresses*. John Wiley, New York, 1960.

TEORIA TERMOELASTICITĂȚII LINIARE A PLĂCILOR

Vibrațiile transversale

(Rezumat)

Se demonstrează o teoremă de unicitate pentru vibrațiile transversale ale plăcilor elastice în cadrul teoriei generalizate a termoelasticității prezentată în [2].

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THE LAGRANGE'S EQUATIONS FOR A RIGID BODY AND GEOMETRICAL INTERPRETATIONS CONCERNING THE PRINCIPLES OF IMPULSE AND OF KINETIC MOMENT

BY

AURORA CRĂCIUNAȘ, ELISABETA RUSU and P. CRĂCIUNAȘ

In this paper one shows that the scalar equations corresponding to the two fundamental principles of the mechanics of rigid body, the „Principle of impulse“ respectively the „Principle of kinetic moment“, can be deduced from the Lagrange's equations but, even if formally the obtained equations are the same, the riemannian geometrical structure connected with the Lagrange's equations is different by the structure of the Euclidean space in which the two principles are formulated.

As it is well known, the principle of impulse states that „the absolute derivative with respect to the time of the impulse equals the resultant of the external forces which act on the rigid“

$$(1) \quad \frac{D\bar{H}}{dt} = \bar{\mathcal{R}}, \quad \frac{D}{dt}(M\bar{v}_c) = \bar{\mathcal{R}},$$

where M is the rigid's mass and $\bar{v}_c$ — the velocity of its mass center.

The scalar equations are

$$(2) \quad \begin{aligned} M_1 \ddot{x}_c \mathcal{R}_1 &=, & M \ddot{x}_{1c} &= 0, \\ M \ddot{x}_{2c} &= \mathcal{R}_2, & \text{respectively} & & (2') \quad M \ddot{x}_{2c} &= 0 \quad \text{for } \bar{\mathcal{R}} = 0, \\ M \ddot{x}_{3c} &= \mathcal{R}_3; & M \ddot{x}_{3c} &= 0 \end{aligned}$$

where x_{1c} , x_{2c} , x_{3c} , represent the coordinates of the rigid's mass center with respect to the absolute cartesian orthogonal frame having the unit vectors $\bar{e}_1$, $\bar{e}_2$, $\bar{e}_3$, the metrical tensor δ_{ij} and the connection coefficients $\Gamma_{jk}^i = 0$.

The principle of the kinetic moment can be stated in the form: „the absolute derivative with respect to the time of the kinetic moment calculated with respect to the rigid's mass center, is equal with the exterior resultant moment calculated with respect to the same mass center“

$$(3) \quad \frac{D\bar{K}_c}{dt} = \bar{\mathcal{M}}_c,$$

with the scalar equations

$$\frac{dK_C^1}{dt} = \mathcal{M}_C^1, \quad \frac{dK_C^2}{dt} = \mathcal{M}_C^2, \quad \frac{dK_C^3}{dt} = \mathcal{M}_C^3,$$

where $\bar{K}_C = K_C^1 \bar{e}_1 + K_C^2 \bar{e}_2 + K_C^3 \bar{e}_3$.

In the basis $\bar{e}_1, \bar{e}_2, \bar{e}_3$, constituted by the unit vectors of centrale axes and connected by $\bar{e}_1, \bar{e}_2, \bar{e}_3$ through the relations

$$(4) \quad \bar{e}_{h'} = \sum_{m=1}^3 a_{h'm} \bar{e}_m \quad \text{with}$$

$$\begin{cases} a_{1'1} = \cos \theta^1 \cos \theta^3 - \sin \theta^1 \cos \theta^2 \sin \theta^3 & a_{1'2} = \sin \theta^1 \cos \theta^3 + \cos \theta^1 \cos \theta^2 \sin \theta^3 & a_{1'3} = \sin \theta^2 \sin \theta^3 \\ a_{2'1} = -\cos \theta^1 \sin \theta^3 - \sin \theta^1 \cos \theta^2 \cos \theta^3 & a_{2'2} = -\sin \theta^1 \sin \theta^3 + \cos \theta^1 \cos \theta^2 \cos \theta^3 & a_{2'3} = \sin \theta^2 \cos \theta^3 \\ a_{3'1} = \sin \theta^1 \sin \theta^2 & a_{3'2} = -\cos \theta^1 \sin \theta^2 & a_{3'3} = \cos \theta^2 \end{cases};$$

the components of the kinetic moment are

$$(5) \quad \begin{aligned} K_C^{1'} &= i_1^2 \omega_{1'} = i_1^2 (\dot{\theta}^1 \sin \theta^2 \cos \theta^3 + \dot{\theta}^2 \sin \theta^3), \\ K_C^{2'} &= i_2^2 \omega_{2'} = i_2^2 (\dot{\theta}^1 \sin \theta^2 \cos \theta^3 - \dot{\theta}^2 \cos \theta^3), \\ K_C^{3'} &= i_3^2 \omega_{3'} = i_3^2 (\dot{\theta}^1 \cos \theta^2 + \dot{\theta}^3), \end{aligned}$$

where i_1^2, i_2^2, i_3^2 , represent the central inertial moments $J^{1'1'} = i_1^2, J^{2'2'} = i_2^2, J^{3'3'} = i_3^2$ and $\theta^1, \theta^2, \theta^3$, the Euler's angles.

Under a change of base (4) the metric remains δ_{ij} while the connection coefficients become

$$\Gamma_{j'k'}^{i'} = -a_{j's} \frac{\partial a_{si'}}{\partial \theta^m} a_{k'm},$$

$$(5') \quad \begin{aligned} \Gamma_{31}^{1'} &= \sin \theta^2 \sin \theta^3, & \Gamma_{32}^{1'} &= -\cos \theta^3, & \Gamma_{21}^{1'} &= -\cos \theta^2, & \Gamma_{23}^{1'} &= -1, \\ \Gamma_{11}^{2'} &= \cos \theta^2, & \Gamma_{13}^{2'} &= 1, & \Gamma_{31}^{2'} &= -\sin \theta^2 \sin \theta^3, & \Gamma_{32}^{2'} &= -\sin \theta^3, \\ \Gamma_{21}^{3'} &= \sin \theta^2 \cos \theta^3, & \Gamma_{22}^{3'} &= \sin \theta^3, & \Gamma_{11}^{3'} &= -\sin \theta^2 \sin \theta^3, & \Gamma_{12}^{3'} &= \cos \theta^3. \end{aligned}$$

The relations $d\bar{e}_{m'}/dt = \Gamma_{m'n'}^{h'} \dot{\theta}^n \bar{e}_{h'}$, represent the well known Poisson's relations

$$\frac{d\bar{e}_{m'}}{dt} = \delta_{m'n'}^{h'} \omega^{n'} \bar{e}_{h'} = \bar{\omega} \times \bar{e}_{m'}.$$

In this base the vectorial equations (3) leads us to the scalar equations

$$(6) \quad \begin{cases} \frac{d}{dt} (i_1^2 \omega_{1'}) + (i_3^2 - i_2^2) \omega_{2'} \omega_{3'} = \mathcal{M}_C^{1'}, \\ \frac{d}{dt} (i_2^2 \omega_{2'}) + (i_1^2 - i_3^2) \omega_{3'} \omega_{1'} = \mathcal{M}_C^{2'}, \\ \frac{d}{dt} (i_3^2 \omega_{3'}) + (i_2^2 - i_1^2) \omega_{1'} \omega_{2'} = \mathcal{M}_C^{3'}, \end{cases}$$

respectively

$$(6') \quad \begin{cases} \frac{d}{dt} (i_1^2 \omega_1) + (i_3^2 - i_2^2) \omega_2 \omega_3 = 0, \\ \frac{d}{dt} (i_2^2 \omega_2) + (i_1^2 - i_3^2) \omega_3 \omega_1 = 0, \text{ for } \overline{M}_C = 0. \\ \frac{d}{dt} (i_3^2 \omega_3) + (i_2^2 - i_1^2) \omega_1 \omega_2 = 0. \end{cases}$$

Let us consider now the so called space of configurations or the Lagrange's space for a rigid body, for which the generalized coordinates are $(x_{1C}, x_{2C}, x_{3C}, \theta^1, \theta^2, \theta^3)$. In this space we define the metric

$$(7) \quad ds^2 = 2\mathcal{E}(dt)^2,$$

where $\mathcal{E}$ represent the kinetic energy of the rigid

$$\mathcal{E} = \frac{1}{2} (M v_C^2 + \bar{K}_C \bar{\omega}) = \frac{1}{2} [M(x_{1C}^2 + x_{2C}^2 + x_{3C}^2) + i_1^2 (\omega_1)^2 + i_2^2 (\omega_2)^2 + i_3^2 (\omega_3)^2].$$

Consequently, so defined ds^2 , depending on the coordinates $(x_{1C}, x_{2C}, x_{3C}, \theta^1, \theta^2, \theta^3)$ has the form

$$(8) \quad \begin{aligned} ds^2 = & M[(dx_{1C})^2 + (dx_{2C})^2 + (dx_{3C})^2] + [\sin^2 \theta^2 (i_1^2 \sin^2 \theta^3 + i_2^2 \cos^2 \theta^3) + \\ & + i_3^2 \cos^2 \theta^2] (d\theta^1)^2 + (i_1^2 \cos^2 \theta^3 + i_2^2 \sin^2 \theta^3) (d\theta^2)^2 + i_3^2 (d\theta^3)^2 + \\ & + 2(i_1^2 - i_2^2) \sin \theta^2 \sin \theta^3 \cos \theta^3 d\theta^1 d\theta^2 + 2i_3^2 \cos \theta^2 d\theta^1 d\theta^3. \end{aligned}$$

Since the equations of the geodesics of this six dimensional riemannian variety V are equivalent with the Lagrange's equations in absence of exterior stress, we will do a more detailed analysis of this geometrical structure.

From (8) it results for the metric tensor the components

$$(9) \quad \begin{array}{ccccccc} g_{11}=1 & g_{12}=0 & g_{13}=0 & 0 & 0 & 0 & 0 \\ g_{21}=0 & g_{22}=1 & g_{23}=0 & 0 & 0 & 0 & 0 \\ g_{31}=0 & g_{32}=0 & g_{33}=1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \tilde{g}_{11}=\sin^2 \theta^2 (i_1^2 \sin^2 \theta^3 + i_2^2 \cos^2 \theta^3) + i_3^2 \cos^2 \theta^2 & \tilde{g}_{12}=(i_1^2 - i_2^2) \sin \theta^2 \sin \theta^3 \cos \theta^3 & \tilde{g}_{13}=\tilde{g}_{31} & \\ 0 & 0 & 0 & \tilde{g}_{21}=\tilde{g}_{12} & \tilde{g}_{22}=i_1^2 \cos^2 \theta^3 + i_2^2 \sin^2 \theta^3 & \tilde{g}_{23}=0 & \\ 0 & 0 & 0 & \tilde{g}_{31}=i_3^2 \cos \theta^2 & \tilde{g}_{32}=0 & \tilde{g}_{33}=\tilde{g}_{33} & \end{array}$$

Taking into account this form of the metric tensor we can consider the variety V as been the reunion of two disjoint subvarieties V_3 generated by the variables x_{1C}, x_{2C}, x_{3C} with the Euclidean metric δ_{ij} and having, like unit vectors of natural base, the unit vectors of the absolute frame $\bar{e}_1, \bar{e}_2, \bar{e}_3$, and the sub-variety V_3' generated by the variables $\theta^1, \theta^2, \theta^3$, having like natural base $(\partial/\partial\theta^1, \partial/\partial\theta^2, \partial/\partial\theta^3)$ defined by

$$(10) \quad \frac{\partial}{\partial\theta^\alpha} \frac{\partial}{\partial\theta^\beta} = \tilde{g}_{\alpha\beta}.$$

For the first subvariety with the metric δ_{ij} , the Cristoffel's symbols are

$$(11) \quad \left| \begin{smallmatrix} i \\ j \ k \end{smallmatrix} \right| = 0 \quad i, j, k = 1, 2, 3.$$

For the subvariety V_3 with the metric $\tilde{g}_{\alpha\beta}$ the Cristoffel's symbols are

$$\left| \begin{smallmatrix} 1 \\ 1 \ 1 \end{smallmatrix} \right| = \frac{(i_2^2 - i_1^2)(i_3^2 - i_1^2 - i_2^2)}{i_1^2 i_2^2} \cos \theta^2 \sin \theta^3 \cos \theta^3,$$

$$\left| \begin{smallmatrix} 1 \\ 1 \ 2 \end{smallmatrix} \right| = \frac{i_1^2 + i_2^2 - i_3^2}{2i_1^2 i_2^2} \frac{\cos \theta^2}{\sin \theta^2} (i_1^2 \cos^2 \theta^3 + i_2^2 \sin^2 \theta^3),$$

$$\left| \begin{smallmatrix} 1 \\ 2 \ 2 \end{smallmatrix} \right| = 0, \quad \left| \begin{smallmatrix} 1 \\ 3 \ 3 \end{smallmatrix} \right| = 0, \quad \left| \begin{smallmatrix} 1 \\ 2 \ 3 \end{smallmatrix} \right| = \frac{-i_1^2(i_2^2 + i_3^2 - i_1^2)(\cos^2 \theta^3 - i_2^2(i_1^2 + i_3^2 - i_2^2)\sin^2 \theta^3)}{2i_1^2 i_2^2 \sin \theta^2},$$

$$\left| \begin{smallmatrix} 1 \\ 1 \ 3 \end{smallmatrix} \right| = \frac{(i_1^2 - i_2^2)(i_1^2 + i_2^2 - i_3^2)}{2i_1^2 i_2^2} \sin \theta^3 \cos \theta^3, \quad \left| \begin{smallmatrix} 2 \\ 2 \ 2 \end{smallmatrix} \right| = 0, \quad \left| \begin{smallmatrix} 2 \\ 3 \ 3 \end{smallmatrix} \right| = 0,$$

$$\left| \begin{smallmatrix} 2 \\ 1 \ 1 \end{smallmatrix} \right| = \frac{\sin \theta^2 \cos \theta^2}{i_1^2 i_2^2} [(i_1^2(i_3^2 - i_1^2) \sin^2 \theta^3 + i_2^2(i_3^2 - i_2^2) \cos^2 \theta^3),$$

$$\left| \begin{smallmatrix} 3 \\ 2 \ 2 \end{smallmatrix} \right| = \frac{i_1^2 - i_2^2}{i_3^2} \sin \theta^3 \cos \theta^3,$$

$$\left| \begin{smallmatrix} 2 \\ 1 \ 3 \end{smallmatrix} \right| = \frac{\sin \theta^2}{2i_1^2 i_2^2} [i_1^2(i_3^2 + i_2^2 - i_1^2) \sin^2 \theta^3 + i_2^2(i_3^2 + i_1^2 - i_2^2) \cos^2 \theta^3],$$

$$(12) \quad \left| \begin{smallmatrix} 2 \\ 2 \ 3 \end{smallmatrix} \right| = \frac{\sin \theta^3 \cos \theta^3}{2i_1^2 i_2^2} (i_1^2 - i_2^2)(i_1^2 + i_2^2 - i_3^2),$$

$$\left| \begin{smallmatrix} 2 \\ 2 \ 1 \end{smallmatrix} \right| = \frac{(i_2^2 - i_1^2)(i_1^2 + i_2^2 - i_3^2)}{2i_1^2 i_2^2} \cos \theta^2 \sin \theta^3 \cos \theta^3,$$

$$\left| \begin{smallmatrix} 3 \\ 1 \ 3 \end{smallmatrix} \right| = \frac{(i_1^2 - i_2^2)(i_3^2 - i_1^2 - i_2^2)}{2i_1^2 i_2^2} \cos \theta^2 \sin \theta^3 \cos \theta^3,$$

$$\left| \begin{smallmatrix} 3 \\ 1 \ 1 \end{smallmatrix} \right| = \frac{i_1^2 - i_2^2}{i_1^2 i_2^2 i_3^2} [i_3^2(i_3^2 - i_1^2 - i_2^2) \cos^2 \theta^2 - i_1^2 i_2^2 \sin^2 \theta^2] \sin \theta^3 \cos \theta^3, \quad \left| \begin{smallmatrix} 3 \\ 3 \ 3 \end{smallmatrix} \right| = 0,$$

$$\left| \begin{smallmatrix} 3 \\ 1 \ 2 \end{smallmatrix} \right| = \frac{\sin \theta^2}{2i_3^2} [(i_2^2 - i_1^2)(\cos^2 \theta^3 - \sin^2 \theta^3) - i_3^2] +$$

$$+ \frac{i_3^2 - i_1^2 - i_2^2}{2i_1^2 i_2^2} \frac{\cos^2 \theta^2}{\sin \theta^2} (i_1^2 \cos^2 \theta^3 + i_2^2 \sin^2 \theta^3),$$

$$\left| \begin{smallmatrix} 3 \\ 2 \ 3 \end{smallmatrix} \right| = \frac{\cos \theta^2}{2i_1^2 i_2^2 \sin \theta^2} [i_1^2(i_3^2 + i_2^2 - i_1^2) \cos^2 \theta^3 + i_2^2(i_3^2 + i_1^2 - i_2^2) \sin^2 \theta^3].$$

The equations of the geodesics for the subvariety V_3 are

$$(13) \quad \frac{d^2 x_{1C}}{ds^2} = 0, \quad \frac{d^2 x_{2C}}{ds^2} = 0, \quad \frac{d^2 x_{3C}}{ds^2} = 0.$$

For the subvariety V_3' these equations are

$$\begin{aligned} \frac{D(d\theta'/ds)}{ds} &= \frac{d^2 \theta^1}{ds^2} + \frac{(i_2^2 - i_1^2)(i_3^2 - i_1^2 - i_2^2)}{i_1^2 i_1^2} \cos \theta^2 \sin \theta^3 \cos \theta^3 \frac{d\theta^1}{ds} \frac{d\theta^1}{ds} + \\ &+ (i_1^2 \cos^2 \theta^3 + i_2^2 \sin^2 \theta^3) \frac{(i_1^2 + i_2^2 - i_3^2) \cos \theta^2 \frac{d\theta^1}{ds} \frac{d\theta^2}{ds}}{i_1^2 i_2^2 \sin \theta^2} + \\ &+ \frac{(i_1^2 - i_2^2)(i_1^2 + i_2^2 - i_3^2)}{i_1^2 i_2^2} \sin \theta^3 \cos \theta^3 \frac{d\theta^1}{ds} \frac{d\theta^3}{ds} + \\ &+ \frac{1}{i_1^2 i_2^2} [i_1^2(i_1^2 - i_2^2 - i_3^2) \cos^2 \theta^3 + i_2^2(i_2^2 - i_1^2 - i_3^2) \sin^2 \theta^3] \frac{1}{\sin \theta^2} \frac{d\theta^2}{ds} \frac{d\theta^3}{ds} = 0, \\ \frac{D(d\theta^2/ds)}{ds} &= \frac{d^2 \theta^2}{ds^2} + \frac{i_1^2(i_3^2 - i_1^2) \sin^2 \theta^3 + i_2^2(i_3^2 - i_2^2) \cos^2 \theta^3}{i_1^2 i_2^2} \sin \theta^2 \cos \theta^2 \frac{d\theta^1}{ds} \frac{d\theta^1}{ds} + \\ &+ \frac{(i_2^2 - i_1^2)(i_1^2 + i_2^2 - i_3^2)}{i_1^2 i_2^2} \cos \theta^2 \sin \theta^3 \cos \theta^3 \frac{d\theta^1}{ds} \frac{d\theta^2}{ds} + \\ (14) \quad &+ \frac{\sin \theta^2}{i_1^2 i_2^2} [i_1^2(i_3^2 + i_2^2 - i_1^2) \sin^2 \theta^3 + i_2^2(i_3^2 + i_1^2 - i_2^2) \cos^2 \theta^3] \frac{d\theta^1}{ds} \frac{d\theta^3}{ds} + \\ &+ \frac{(i_2^2 - i_1^2)(i_1^2 + i_2^2 - i_3^2)}{i_1^2 i_2^2} \sin \theta^3 \cos \theta^3 \frac{d\theta^1}{ds} \frac{d\theta^3}{ds} = 0, \\ \frac{D(d\theta^3/ds)}{ds} &= \frac{d^2 \theta^3}{ds^2} + \frac{i_2^2 - i_1^2}{i_1^2 i_2^2} \sin \theta^3 \cos \theta^3 \left[(i_3^2 - i_1^2 - i_2^2) \cos^2 \theta^2 - \right. \\ &- \left. \frac{i_1^2 i_2^2}{i_3^2} \sin^2 \theta^2 \right] \frac{d\theta^1}{ds} \frac{d\theta^1}{ds} + \frac{i_1^2 - i_2^2}{i_3^2} \sin \theta^3 \cos \theta^3 \frac{d\theta^2}{ds} \frac{d\theta^2}{ds} + \\ &+ \left\{ \frac{\sin \theta^2}{i_3^2} [(i_2^2 - i_1^2)(\cos^2 \theta^3 - \sin^2 \theta^3) - i_3^2] + \right. \\ &+ \left. \frac{i_3^2 - i_1^2 - i_2^2}{i_1^2 i_2^2} (i_1^2 \cos^2 \theta^3 + i_2^2 \sin^2 \theta^3) \frac{\cos^2 \theta^2}{\sin^2 \theta^2} \right\} \frac{d\theta^1}{ds} \frac{d\theta^2}{ds} + \\ &+ \frac{(i_3^2 - i_1^2 - i_2^2)(i_1^2 - i_2^2)}{i_1^2 i_2^2} \cos \theta^2 \cos \theta^3 \sin \theta^3 \frac{d\theta^1}{ds} \frac{d\theta^3}{ds} + \\ &+ \frac{\cos \theta^2}{i_1^2 i_2^2 \sin \theta^3} [i_1^2(i_3^2 + i_2^2 - i_1^2) \cos^2 \theta^3 + i_2^2(i_3^2 + i_1^2 - i_2^2) \sin^2 \theta^3] \frac{d\theta^2}{ds} \frac{d\theta^3}{ds} = 0. \end{aligned}$$

Let us suppose that the time variable t is connected by ds through the relation

$$(15) \quad ds = \mathcal{L}_0 dt.$$

From a mechanic point of view, (15) means to define the time variable t such that, in absence of exterior stress, the motion be made with the conservation of kinetic rotation energy. The time variable with respect to which are written the relations (2') and (6') satisfies the imposed condition (15) i.e. $\mathcal{L} = \mathcal{L}_0$.

Considering (15) like a change variables in the Eqs. (13) we see that these equations coincide with (2') from the principle of impulse.

It is less visible the fact that Eqs. (14), in the same condition $\mathcal{L} = \mathcal{L}_0$, coincide with Eq. (6') of the principle of kinetic moment.

To show this we will pay attention to the variety V_3 where we make the change of basis

$$(16) \quad \begin{bmatrix} \frac{\partial}{\partial \eta^{1'}} \\ \frac{\partial}{\partial \eta^{2'}} \\ \frac{\partial}{\partial \eta^{3'}} \end{bmatrix} = \begin{bmatrix} \mu_1^1 = \frac{\sin \theta^3}{i_1^2 \sin \theta^2} & \mu_1^2 = \frac{\cos \theta^3}{i_2^2 \sin \theta^2} & \mu_1^3 = 0 \\ \mu_2^1 = \frac{\cos \theta^3}{i_1^2} & \mu_2^2 = \frac{-\sin \theta^3}{i_2^2} & \mu_2^3 = 0 \\ \mu_3^1 = \frac{-\sin \theta^3 \cos \theta^2}{i_1^2 \sin \theta^2} & \mu_3^2 = \frac{-\cos \theta^2 \cos \theta^3}{i_2^2 \sin \theta^2} & \mu_3^3 = \frac{1}{i_3^2} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \theta^1} \\ \frac{\partial}{\partial \theta^2} \\ \frac{\partial}{\partial \theta^3} \end{bmatrix}.$$

As a result of this change change of basis, the coordinates $\dot{\theta}^1, \dot{\theta}^2, \dot{\theta}^3$ of the vector $\dot{\theta}^1 \partial / \partial \theta^1 + \dot{\theta}^2 \partial / \partial \theta^2 + \dot{\theta}^3 \partial / \partial \theta^3$ become

$$\begin{bmatrix} i_1^2 \omega_{1'} \\ i_2^2 \omega_{2'} \\ i_3^2 \omega_{3'} \end{bmatrix} = \begin{bmatrix} \lambda_1^1 = i_1^2 \sin \theta^3 \cos \theta^2 & \lambda_1^2 = i_1^2 \sin \theta^3 & \lambda_1^3 = 0 \\ \lambda_2^1 = i_2^2 \sin \theta^3 \sin \theta^2 & \lambda_2^2 = -i_2^2 \cos \theta^3 & \lambda_2^3 = 0 \\ \lambda_3^1 = 0 & \lambda_3^2 = 0 & \lambda_3^3 = i_3^2 \end{bmatrix} \begin{bmatrix} \dot{\theta}^1 \\ \dot{\theta}^2 \\ \dot{\theta}^3 \end{bmatrix}.$$

We have hence

$$\dot{\theta}^1 \frac{\partial}{\partial \theta^1} + \dot{\theta}^2 \frac{\partial}{\partial \theta^2} + \dot{\theta}^3 \frac{\partial}{\partial \theta^3} = i_1^2 \omega_{1'} \frac{\partial}{\partial \eta^{1'}} + i_2^2 \omega_{2'} \frac{\partial}{\partial \eta^{2'}} + i_3^2 \omega_{3'} \frac{\partial}{\partial \eta^{3'}},$$

where $i_1^2 \omega_{1'}$, $i_2^2 \omega_{2'}$, $i_3^2 \omega_{3'}$, represent the component of the kinetic moment with respect to the central axis, in the basis $\partial / \partial \eta^{1'}$, $\partial / \partial \eta^{2'}$, $\partial / \partial \eta^{3'}$.

To transcribe, in this basis the geodesics equations (14), let calculate the new connexion coefficients. To the change of basis (16) we will have

$$(17) \quad \gamma_{b'c'}^{a'} = \left(\begin{matrix} r \\ i \ j \end{matrix} \middle| \lambda_{r'}^{a'} - \frac{\partial \lambda_{i'}^{a'}}{\partial \theta^j} \right) \mu_{b'}^i \mu_{c'}^j.$$

If

$$A_{ij}^{a'} = \left(\begin{matrix} r \\ i \ j \end{matrix} \middle| \lambda_{r'}^{a'} - \frac{\partial \lambda_{i'}^{a'}}{\partial \theta^j} \right),$$

one obtains

$$\begin{aligned}
 A_{11}' &= (i_3^2 - i_2^2) \sin \theta^2 \cos \theta^2 \cos \theta^3, & A_{22}' &= 0, \\
 A_{12}' &= \frac{1}{2}(i_2^2 - i_1^2 - i_3^2) \sin \theta^3 \cos \theta^2, & A_{33}' &= 0, \\
 A_{21}' &= \frac{1}{2}(i_1^2 + i_2^2 + i_3^2) \sin \theta^3 \cos \theta^2, & A_{31}' &= \frac{1}{2}(i_1^2 + i_3^2 - i_2^2) \sin \theta^2 \cos \theta^3, \\
 A_{13}' &= \frac{1}{2}(i_3^2 - i_1^2 - i_2^2) \sin \theta^2 \cos \theta^3, & A_{23}' &= \frac{1}{2}(i_2^2 + i_1^2 - i_3^2) \sin \theta^3, \\
 A_{32}' &= \frac{1}{2}(i_2^2 - i_3^2 - i_1^2) \sin \theta^3, & A_{22}' &= 0, \\
 A_{11}^{2'} &= (i_1^2 - i_3^2) \sin \theta^2 \cos \theta^2 \sin \theta^3, & A_{33}^{2'} &= 0, \\
 A_{12}^{2'} &= \frac{1}{2}(i_1^2 + i_2^2 - i_3^2) \cos \theta^2 \cos \theta^3, & A_{21}^{2'} &= \frac{1}{2}(i_1^2 + i_2^2 - i_3^2) \cos \theta^2 \cos \theta^3, \\
 A_{31}^{2'} &= \frac{1}{2}(i_1^2 - i_2^2 - i_3^2) \sin \theta^2 \sin \theta^3, & A_{13}^{2'} &= \frac{1}{2}(i_1^2 + i_2^2 - i_3^2) \sin \theta^2 \sin \theta^3, \\
 A_{23}^{2'} &= \frac{1}{2}(i_1^2 + i_2^2 - i_3^2) \cos \theta^3, & A_{32}^{2'} &= \frac{1}{2}(i_1^2 - i_2^2 - i_3^2) \cos \theta^3, \\
 A_{11}^{3'} &= (i_2^2 - i_1^2) \sin^2 \theta^2 \sin \theta^3 \cos \theta^3, & A_{22}^{3'} &= (i_1^2 - i_2^2) \sin \theta^3 \cos \theta^3, \\
 A_{12}^{3'} &= \frac{1}{2} \sin \theta^2 [(i_2^2 - i_1^2)(\cos^2 \theta^3 - \sin^2 \theta^3) + i_3^2], & A_{23}^{3'} &= 0, \quad A_{13}^{3'} = 0, \\
 A_{21}^{3'} &= \frac{1}{2} \sin \theta^2 [(i_2^2 - i_1^2)(\cos^2 \theta^3 - \sin^2 \theta^3) - i_3^2], & A_{32}^{3'} &= 0, \quad A_{31}^{3'} = 0.
 \end{aligned}$$

Substituting in (17) we find as being different of zero the following connexion coefficients

$$\begin{aligned}
 \gamma_{23}^{1'} &= \frac{i_3^2 - i_1^2 - i_2^2}{2i_2^2 i_3^2}, & \gamma_{3'2}^{1'} &= \frac{i_3^2 + i_1^2 - i_2^2}{2i_2^2 i_3^2}, \\
 \gamma_{3'1}^{2'} &= \frac{i_1^2 - i_3^2 - i_2^2}{2i_1^2 i_3^2}, & \gamma_{1'3}^{2'} &= \frac{i_1^2 + i_3^2 - i_2^2}{2i_1^2 i_3^2}, \\
 \gamma_{1'2'}^{3'} &= \frac{i_2^2 - i_1^2 - i_3^2}{2i_1^2 i_2^2}, & \gamma_{2'1'}^{3'} &= \frac{i_2^2 + i_3^2 - i_1^2}{2i_1^2 i_2^2}.
 \end{aligned}
 \tag{18}$$

In the new basis the components $\tilde{g}_{\alpha\beta}$ of the metric tensor from (9) become

$$[\tilde{g}'_{\alpha\beta}] = \begin{bmatrix} \frac{1}{i_1^2} & 0 & 0 \\ 0 & \frac{1}{i_2^2} & 0 \\ 0 & 0 & \frac{1}{i_3^2} \end{bmatrix},$$

the contravariant components $\tilde{g}'^{\alpha\beta}$ being

$$[\tilde{g}'^{\alpha\beta}] = \begin{bmatrix} i_1^2 & 0 & 0 \\ 0 & i_2^2 & 0 \\ 0 & 0 & i_3^2 \end{bmatrix},$$

The space curvature, easier to be calculated starting from the form (18) of the connection coefficients, is

$$K = \frac{1}{2i_1^2 i_2^2 i_3^2} (i_1^2 + i_2^2 + i_3^2)(i_1^2 + i_2^2 - i_3^2)(i_2^2 + i_3^2 - i_1^2)(i_2^2 + i_3^2 - i_1^2).$$

In the basis $\partial/\partial\eta^1, \partial/\partial\eta^2, \partial/\partial\eta^3$, the geodesics equations are

$$\frac{D}{dt} \dot{\eta}^{h'} = \frac{d}{dt} (\dot{\eta}^{h'}) + \gamma_{b'c'}^{h'} \dot{\eta}^{b'} \dot{\eta}^{c'} = 0,$$

where $\eta^{h'} = i_k^2 \omega_{h'} = J^{h'h} \omega_{h'}$ (without summation on h').

Consequently, in the place of Eqs. (14) we have

$$\begin{aligned} \frac{D}{dt} (\dot{\eta}^1) &= \frac{d}{dt} (i_1^2 \omega_1) + (i_3^2 - i_2^2) \omega_2 \omega_3 = 0, \\ \frac{D}{dt} (\dot{\eta}^2) &= \frac{d}{dt} (i_2^2 \omega_2) + (i_1^2 - i_3^2) \omega_1 \omega_3 = 0, \\ \frac{D}{dt} (\dot{\eta}^3) &= \frac{d}{dt} (i_3^2 \omega_3) + (i_2^2 - i_1^2) \omega_1 \omega_2 = 0. \end{aligned} \quad (19)$$

One observes that the Eqs. (19) coincide with (6') of the principle of kinetic moment.

The Lagrange's equations for a rigid body, in absence of exterior stress, are

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{x}_{hc}} \right) - \frac{\partial \mathcal{L}}{\partial x_{hc}} = 0, \quad h=1, 2, 3, \quad (20)$$

$$\frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}^\alpha} \right) - \frac{\partial \mathcal{L}}{\partial \theta^\alpha} = 0, \quad \alpha=1, 2, 3. \quad (21)$$

The Eqs. (20) coincide with (13) and with (2') while (21), as it is well known, does not give us anything than the covariant derivatives of the covariant components

$$\dot{\theta}_\alpha = \tilde{g}_{\alpha\beta} \dot{\theta}^\beta. \quad (22)$$

Therefore, if the Eqs. (14) represent, in the structure of the variety V'_3 , $D\dot{\theta}^\alpha/dt=0$, $\alpha=1, 2, 3$, the Eqs. (21) can be written as

$$\frac{D\dot{\theta}_\alpha}{dt} = 0, \quad \alpha=1, 2, 3. \quad (23)$$

With the purpose of a complete writting of the Lagrange's equations for a rigid body subjected to exterior stress having the resultant $\bar{\mathcal{R}} = \mathcal{R}_1 \bar{e}_1 + \mathcal{R}_2 \bar{e}_2 + \mathcal{R}_3 \bar{e}_3$ and the resultant moment $\bar{\mathcal{M}}_C = \mathcal{M}_C^1 \bar{e}_1 + \mathcal{M}_C^2 \bar{e}_2 + \mathcal{M}_C^3 \bar{e}_3$, let calculate the components of the generalized forces ($Q_1, Q_2, Q_3, \tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3$), after having specified what we understand through a possible displacement in the case of a rigid body.

The quality of rigid body imposes the condition that each of its points be given by three coordinates (x^1, x^2, x^3). In our case they are the coordinates of a point with respect to the central axis.

The time dependence of the point position of a rigid body is expressed through the six parameters ($x_{1C}, x_{2C}, x_{3C}, \theta^1, \theta^2, \theta^3$). In this way the velocity of a point of the rigid body has the expression

$$\vec{v} = \dot{x}_{1C}\vec{e}_1 + \dot{x}_{2C}\vec{e}_2 + \dot{x}_{3C}\vec{e}_3 + \begin{vmatrix} \vec{e}^1 & \vec{e}^2 & \vec{e}^3 \\ \dot{\theta}^1 \sin \theta^2 \sin \theta^3 + \dot{\theta}^2 \cos \theta^3 & \dot{\theta}^1 \sin \theta^2 \cos \theta^3 - \dot{\theta}^2 \sin \theta^3 & \dot{\theta}^1 \cos \theta^2 + \dot{\theta}^3 \\ x^1 & x^2 & x^3 \end{vmatrix}.$$

An other possible velocity of the same point of the rigid will be

$$\vec{v}' = \dot{\tilde{x}}_{1C}\vec{e}_1 + \dot{\tilde{x}}_{2C}\vec{e}_2 + \dot{\tilde{x}}_{3C}\vec{e}_3 + \begin{vmatrix} \vec{e}^1 & \vec{e}^2 & \vec{e}^3 \\ \tilde{\theta}^1 \sin \theta^2 \sin \theta^3 + \tilde{\theta}^2 \cos \theta^3 & \tilde{\theta}^1 \sin \theta^2 \cos \theta^3 - \tilde{\theta}^2 \sin \theta^3 & \tilde{\theta}^1 \cos \theta^2 + \tilde{\theta}^3 \\ x^1 & x^2 & x^3 \end{vmatrix}.$$

In these two expressions, different values have the variables ($\dot{x}_{1C}, \dot{x}_{2C}, \dot{x}_{3C}, \dot{\theta}^1, \dot{\theta}^2, \dot{\theta}^3$) respectively ($\dot{\tilde{x}}_{1C}, \dot{\tilde{x}}_{2C}, \dot{\tilde{x}}_{3C}, \tilde{\theta}^1, \tilde{\theta}^2, \tilde{\theta}^3$).

Using the Gantmaher definition of virtual displacements as being the difference between two possible infinitesimal displacement, we have

$$(24) \quad \delta \vec{r} = (\vec{v} - \vec{v}') dt = \delta x_{1C} \vec{e}_1 + \delta x_{2C} \vec{e}_2 + \delta x_{3C} \vec{e}_3 + \delta \vec{\psi} \times \vec{r},$$

where $\delta \vec{\psi}$ denotes

$$(25) \quad \delta \vec{\psi} = (\sin \theta^2 \sin \theta^3 \delta \theta^1 + \cos \theta^3 \delta \theta^2) \vec{e}^1 + (\sin \theta^2 \cos \theta^3 \delta \theta^1 - \sin \theta^3 \delta \theta^2) \vec{e}^2 + (\cos \theta^2 \delta \theta^1 + \delta \theta^3) \vec{e}^3.$$

Consequently, independently of the manner in which is given the exterior stress distribution, one obtains

$$\delta \mathcal{L} = \mathcal{R}_1 \delta x_{1C} + \mathcal{R}_2 \delta x_{2C} + \mathcal{R}_3 \delta x_{3C} + (\mathcal{M}^1 \sin \theta^2 \sin \theta^3 + \mathcal{M}^2 \sin \theta^2 \cos \theta^3 + \mathcal{M}^3 \cos \theta^2) \delta \theta^1 + (\mathcal{M}^1 \cos \theta^3 - \mathcal{M}^2 \sin \theta) \delta \theta^2 + \mathcal{M}^3 \delta \theta^3;$$

the components of the exterior force are

$$(26) \quad \begin{aligned} Q_1 &= \mathcal{R}_1, & Q_2 &= \mathcal{R}_2, & Q_3 &= \mathcal{R}_3; \\ \tilde{Q}_1 &= \mathcal{M}^1 \sin \theta^2 \sin \theta^3 + \mathcal{M}^2 \sin \theta^2 \cos \theta^3 + \mathcal{M}^3 \cos \theta^2, \\ \tilde{Q}_2 &= \mathcal{M}^1 \cos \theta^3 - \mathcal{M}^2 \sin \theta, \\ \tilde{Q}_3 &= \mathcal{M}^3. \end{aligned}$$

The Lagrange's equations for the rigid body are

$$(27) \quad M \ddot{x}_{1C} = \mathcal{R}_1, \quad M \ddot{x}_{2C} = \mathcal{R}_2, \quad M \ddot{x}_{3C} = \mathcal{R}_3,$$

respectively

$$(27') \quad \frac{d}{dt} \left(\frac{\partial \mathcal{L}}{\partial \dot{\theta}^\alpha} \right) - \frac{\partial \mathcal{L}}{\partial \theta^\alpha} = \tilde{Q}_\alpha, \quad \alpha = 1, 2, 3,$$

where (27') has the signification

$$(28) \quad \frac{D\dot{\theta}_1}{dt} = \tilde{Q}_1, \quad \frac{D\dot{\theta}_2}{dt} = \tilde{Q}_2, \quad \frac{D\dot{\theta}_3}{dt} = \tilde{Q}_3.$$

To transcribe the Eqs. (28) in contravariant coordinates, using the relation

$$(29) \quad \frac{D\dot{\theta}^\alpha}{dt} = \tilde{Q}^\alpha, \quad \alpha = 1, 2, 3,$$

we will obtain

$$(30) \quad \frac{D\dot{\theta}^\alpha}{dt} = \frac{D}{dt} (g^{\alpha\beta} \dot{\theta}_\beta) = g^{\alpha\beta} \frac{D\dot{\theta}_\beta}{dt} = g^{\alpha\beta} \tilde{Q}_\beta.$$

Since the left member of (30) is given in (14), let calculate $\tilde{Q}^\alpha = g^{\alpha\beta} \tilde{Q}_\beta$, where $g^{\alpha\beta}$ represent the components of the inverse of the matrix $(g_{\alpha\beta})$ given in (9). One obtains

$$1) \quad \begin{bmatrix} \tilde{Q}^1 \\ \tilde{Q}^2 \\ \tilde{Q}^3 \end{bmatrix} = \begin{bmatrix} \frac{i_1^2 \cos^2 \theta^3 + i_2^2 \sin^2 \theta^3}{i_1^2 i_2^2 \sin^2 \theta^2} & \frac{(i_2^2 - i_1^2) \sin \theta^3 \cos \theta^3}{i_1^2 i_2^2 \sin \theta^2} & -\frac{\cos \theta^2 (i_1^2 \cos^2 \theta^3 + i_2^2 \sin^2 \theta^3)}{i_1^2 i_2^2 \sin \theta^2} \\ \frac{(i_2^2 - i_1^2) \sin \theta^3 \cos \theta^3}{i_1^2 i_2^2 \sin \theta^2} & \frac{i_1^2 \sin^2 \theta^3 + i_2^2 \cos^2 \theta^3}{i_1^2 i_2^2} & \frac{(i_1^2 - i_2^2) \sin \theta^3 \cos \theta^3 \cos \theta^2}{i_1^2 i_2^2 \sin \theta^2} \\ -\frac{\cos \theta^2 (i_2^2 \sin^2 \theta^3 + i_1^2 \cos^2 \theta^3)}{i_1^2 i_2^2 \sin^2 \theta^2} & \frac{(i_1^2 - i_2^2) \sin \theta^3 \cos \theta^3 \cos \theta^2}{i_1^2 i_2^2 \sin \theta^2} & \frac{1}{i_2^2} + \frac{\cos^2 \theta^2 (i_1^2 \sin^2 \theta^3 + i_2^2 \cos^2 \theta^3)}{i_1^2 i_2^2 \sin^2 \theta^2} \end{bmatrix}$$

$$\begin{bmatrix} \mathcal{M}_1' \sin \theta^2 \sin \theta^3 + \mathcal{M}_2' \sin \theta^2 + \mathcal{M}_3' \cos \theta^2 \\ \mathcal{M}_1' \cos \theta^3 - \mathcal{M}_2' \sin \theta^3 \\ \mathcal{M}_3' \end{bmatrix} = \begin{bmatrix} \frac{\sin \theta^3}{i_1^2 \sin \theta^2} \mathcal{M}_1' + \frac{\cos \theta^3}{i_2^2 \sin \theta^2} \mathcal{M}_2' \\ \frac{\cos \theta^3}{i_1^2} \mathcal{M}_1' - \frac{\sin \theta^3}{i_2^2} \mathcal{M}_2' \\ -\frac{\cos \theta^2 \sin \theta^3}{i_1^2 \sin \theta^2} \mathcal{M}_1' - \frac{\cos \theta^2 \cos \theta^3}{i_2^2 \sin \theta^2} \mathcal{M}_2' + \frac{1}{i_2^2} \mathcal{M}_3' \end{bmatrix}.$$

In this way Eq. (30) with the left hand member given by (14) have the form

$$(32) \quad \begin{aligned} \frac{D\dot{\theta}^1}{dt} &= \mathcal{M}_1' \frac{\sin \theta^3}{i_1^2 \sin \theta^2} + \mathcal{M}_2' \frac{\cos \theta^3}{i_2^2 \sin \theta^2}, \\ \frac{D\dot{\theta}^2}{dt} &= \mathcal{M}_1' \frac{\cos \theta^3}{i_1^2} - \mathcal{M}_2' \frac{\sin \theta^3}{i_2^2}, \\ \frac{D\dot{\theta}^3}{dt} &= -\mathcal{M}_1' \frac{\cos \theta^2 \sin \theta^3}{i_1^2 \sin \theta^2} - \mathcal{M}_2' \frac{\cos \theta^2 \cos \theta^3}{i_2^2 \sin \theta^2} + \mathcal{M}_3' \frac{1}{i_2^2}. \end{aligned}$$

Let us make the change of basis (16) which gives for the left member from (32), the relations (19). As a result of this change of basis we will have

$$\begin{bmatrix} \lambda_1^{1'} = i^2 \sin \theta^2 \cos \theta^3 & \lambda_2^{1'} = i^2 \sin \theta^3 & 0 \\ \lambda_1^{2'} = i_2^2 \sin \theta^2 \sin \theta^3 & \lambda_2^{2'} = -i_2^2 \cos \theta^3 & 0 \\ 0 & 0 & \lambda_3^{3'} = i_3^2 \end{bmatrix} \cdot \begin{bmatrix} \tilde{Q}^1 \\ \tilde{Q}^2 \\ \tilde{Q}^3 \end{bmatrix} = \begin{bmatrix} \mathcal{M}^{1'} \\ \mathcal{M}^{2'} \\ \mathcal{M}^{3'} \end{bmatrix},$$

so that the Lagrange's equations (28) receive the form

$$\begin{aligned} \frac{d}{dt} (i_1^2 \omega_{1'}) + (i_3^2 - i_2^2) \omega_2 \omega_{3'} &= \mathcal{M}_0^{1'}, \\ \frac{d}{dt} (i_2^2 \omega_{2'}) + (i_1^2 - i_3^2) \omega_1 \omega_{3'} &= \mathcal{M}_0^{2'}, \\ \frac{d}{dt} (i_3^2 \omega_{3'}) + (i_2^2 - i_1^2) \omega_1 \omega_{2'} &= \mathcal{M}_0^{3'} \end{aligned} \quad (33)$$

and they coincide with Eqs. (6) deduced from the principle of kinetic moment.

The Eqs. (33) have been obtained in a general structure different of that of a Euclidean space where they were deduced from the principle of kinetic moment. Formally the Eqs (33) and (6), as well as their solutions, coincide.

Depending on the geometric structure in which we postulate the equations (6), respectively we deduce them from the Lagrange's equations, we have different geometric interpretations. So, in the case of the Eqs. (6) we will say that the motion is made under the action of a constant couple if $\mathcal{M}_0^{1'}$, $\mathcal{M}_0^{2'}$, $\mathcal{M}_0^{3'}$ satisfy the equations

$$\begin{aligned} \frac{d}{dt} (\mathcal{M}_0^{1'}) + \omega^{2'} \mathcal{M}_0^{3'} - \omega^{3'} \mathcal{M}_0^{2'} &= 0, \\ \frac{d}{dt} (\mathcal{M}_0^{2'}) + \omega^{3'} \mathcal{M}_0^{1'} - \omega^{1'} \mathcal{M}_0^{3'} &= 0, \\ \frac{d}{dt} (\mathcal{M}_0^{3'}) + \omega^{1'} \mathcal{M}_0^{2'} - \omega^{2'} \mathcal{M}_0^{1'} &= 0. \end{aligned} \quad (34)$$

The same components $\mathcal{M}_0^{1'}$, $\mathcal{M}_0^{2'}$, $\mathcal{M}_0^{3'}$ will be considered constant in the structure of the variety V_3 for (33) if they satisfy the equations

$$\begin{aligned} \frac{d}{dt} (\mathcal{M}_0^{1'}) + \frac{i_3^2 + i_1^2 - i_2^2}{2i_2^2 i_3^2} i_2^2 \omega_2 \mathcal{M}_0^{3'} + \frac{i_3^2 - i_2^2 - i_1^2}{2i_2^2 i_3^2} \mathcal{M}_0^{2'} i_2^2 \omega_{3'} &= 0, \\ \frac{d}{dt} (\mathcal{M}_0^{2'}) + \frac{i_1^2 + i_2^2 - i_3^2}{2i_1^2} \omega_3 \mathcal{M}_0^{1'} + \frac{i_1^2 - i_3^2 - i_2^2}{2i_3^2} \omega_1 \mathcal{M}_0^{3'} &= 0, \\ \frac{d}{dt} (\mathcal{M}_0^{3'}) + \frac{i_2^2 + i_3^2 - i_1^2}{2i_2^2} \omega_1 \mathcal{M}_0^{2'} + \frac{i_2^2 - i_1^2 - i_3^2}{2i_1^2} \omega_2 \mathcal{M}_0^{1'} &= 0. \end{aligned} \quad (35)$$

The solutions of the Eqs. (34) and (35) are evidently distinct.

Let us observe that in the structure of variety V'_3 a vector having the constant coordinates x^1, x^2, x^3 does not conserve its euclidean length given by $l^2 = (x^1)^2 + (x^2)^2 + (x^3)^2$.

Indeed, calculating one obtains

$$\begin{aligned}
 \frac{D}{dt}(l^2) &= 2x^1 \frac{D}{dt}(x^1) + 2x^2 \frac{D}{dt}(x^2) + 2x^3 \frac{D}{dt}(x^3) = 2x^1 \left(\frac{d}{dt} x^1 + \right. \\
 (36) \quad &+ \gamma_{2'3'}^{1'} x^{2'} \eta^{3'} + \gamma_{3'2'}^{1'} x^{3'} \eta^{2'} \Big) + 2x^2 \left(\frac{d}{dt} x^2 + \gamma_{3'1'}^{2'} x^{3'} \eta^{1'} + \gamma_{1'3'}^{2'} x^{1'} \eta^{3'} \right) + \\
 &+ 2x^3 \left(\frac{d}{dt} x^3 + \gamma_{1'2'}^{3'} x^{1'} \eta^{2'} + \gamma_{2'1'}^{3'} x^{2'} \eta^{1'} \right) = \frac{d}{dt} [(x^1)^2 + (x^2)^2 + (x^3)^2] + \\
 &+ \frac{(i_2^2 - i_1^2)(i_1^2 + i_2^2 - i_3^2)}{i_1^2 i_2^2} x^1 x^2 \omega_3 + \frac{(i_3^2 - i_2^2)(i_2^2 + i_3^2 - i_1^2)}{i_2^2 i_3^2} x^2 x^3 \omega_1 + \\
 &+ \frac{(i_1^2 - i_3^2)(i_3^2 + i_1^2 - i_2^2)}{i_1^2 i_3^2} x^1 x^3 \omega_2;
 \end{aligned}$$

we observe that $\frac{D}{dt}(l^2) \neq \frac{d}{dt}(l^2)$, relation which implies the fact that, in the structure of the riemannian variety V'_3 , the euclidean distance is not an invariant.

The relation (36) clears the fact that $\delta_{\alpha\beta}$ are not invariant by parallel transport on this variety, V'_3 .

In this sense the calculus in (36) should be made starting by $l^2 = \delta_{\alpha'\beta'} x^{\alpha'} x^{\beta'}$ which implies

$$\frac{D}{dt}(l^2) = \delta_{\alpha'\beta'} x^{\alpha'} \frac{D}{dt}(x^{\beta'}) + x^{\alpha'} x^{\beta'} \frac{D}{dt}(\delta_{\alpha'\beta'}) = \frac{d}{dt}(l^2).$$

We will write the derivatives $\frac{D}{dt} \delta_{\alpha\beta}$ as follows :

$$\begin{aligned}
 \frac{D}{dt} \delta_{12} &= -\gamma_{13}^2 \eta^3 - \gamma_{32}^1 \eta^3 = -\frac{(i_2^2 - i_1^2)(i_1^2 + i_2^2 - i_3^2)}{2i_1^2 i_2^2} \omega_2, \\
 (37) \quad \frac{D}{dt} \delta_{23} &= -\gamma_{21}^3 \eta^1 - \gamma_{31}^2 \eta^1 = -\frac{(i_3^2 - i_2^2)(i_2^2 + i_3^2 - i_1^2)}{2i_2^2 i_3^2} \omega_1, \\
 \frac{D}{dt} \delta_{31} &= -\gamma_{32}^1 \eta^2 - \gamma_{12}^3 \eta^2 = -\frac{(i_1^2 - i_3^2)(i_3^2 + i_1^2 - i_2^2)}{2i_1^2 i_3^2} \omega_2.
 \end{aligned}$$

If we consider, in this variety V'_3 , the unit vectors of the central axis $\bar{e}_1, \bar{e}_2, \bar{e}_3$, satisfying $\bar{e}_\alpha \bar{e}_\beta = \delta_{\alpha\beta}$, based on the relations (37) we have

$$\begin{aligned}
 \frac{D}{dt}(\bar{e}_1, \bar{e}_2) &= \frac{(i_1^2 - i_2^2)(i_1^2 + i_2^2 - i_3^2)}{2i_1^2 i_2^2} \omega_3, & \frac{D}{dt}(\bar{e}_2, \bar{e}_3) &= \frac{(i_2^2 - i_3^2)(i_2^2 + i_3^2 - i_1^2)}{2i_2^2 i_3^2} \omega_1, \\
 \frac{D}{dt}(\bar{e}_3, \bar{e}_1) &= \frac{(i_3^2 - i_1^2)(i_1^2 + i_3^2 - i_2^2)}{2i_1^2 i_3^2} \omega_2.
 \end{aligned}$$

Therefore the angle between the unit vectors of central axis is variable in time on V_3 .

In conclusion, the six equations representing the principle of impulse, respectively the principle of kinetic moment for a rigid body, can be obtained from the Lagrange's equations. But, if the first three of them can be considered as been deduced in a three dimensional euclidean subvariety of the space V_6 generated by the variables $(x_{1C}, x_{2C}, x_{3C}; \theta^1, \theta^2, \theta^3)$, the other three Eqs. (33) are deduced in a subvariety of V_6 in whose geometric structure the perpendicularity and the euclidean distance are not preserved.

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REFERENCES

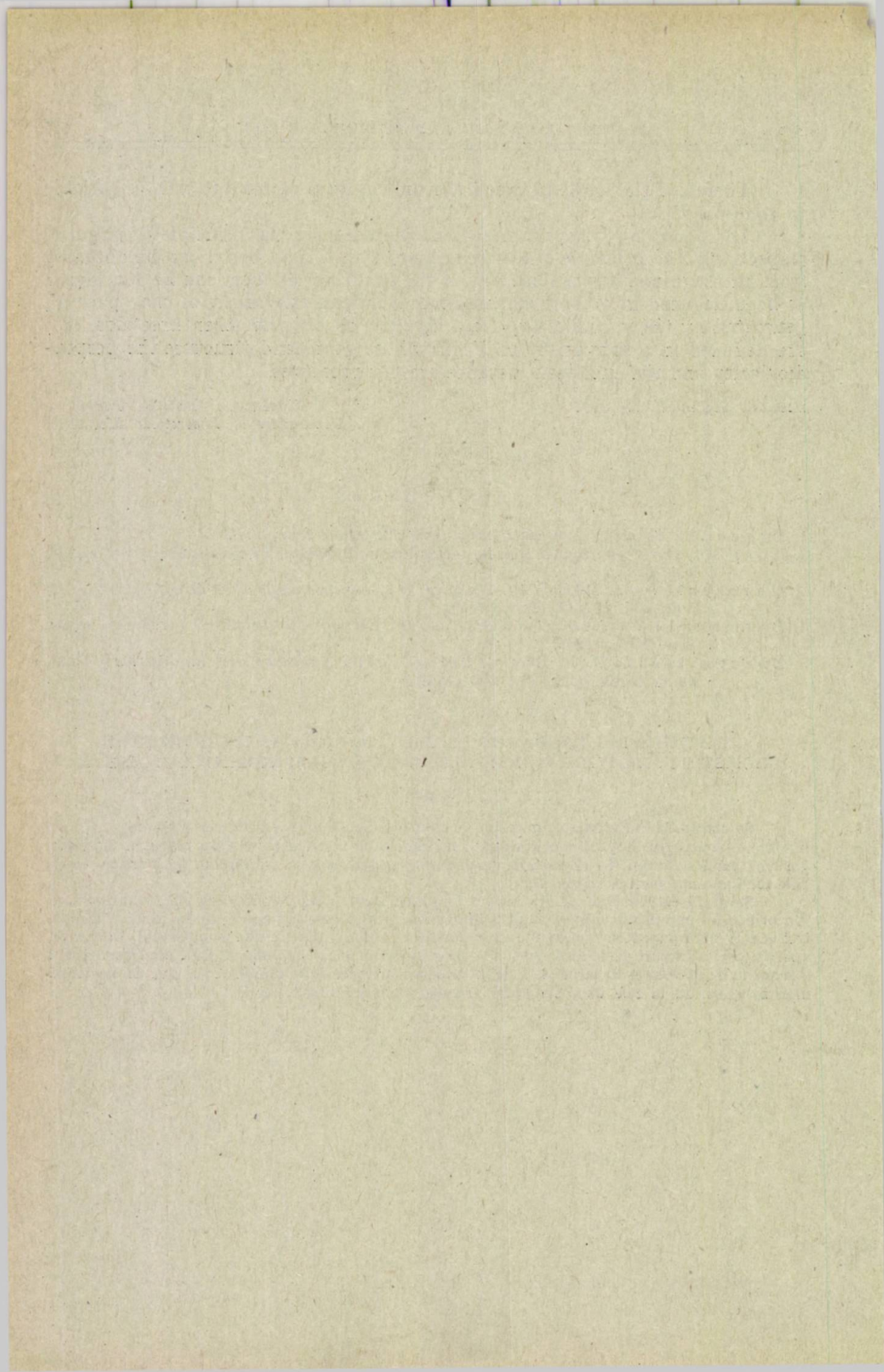
1. Dragoș L., *Principiile mecanicii analitice*. Ed. tehn., Buc., 1976.
2. Borș C. I., *Lecții de mecanică*. Vol. II, Centrul de multiplicare al Universității „Al. I. Cuza”, Iași, 1987.
3. Gheorghiev Gh., Miron R., Papuc D., *Geometrie analitică și diferențială*. Vol. II, Edit. did. și ped., Buc., 1969.
4. Hermann R., *Differential Geometry and the Calculus of Variations*. Academic Press, New York, 1968.
5. Boceyavleeski O. J., *Integrale Cases of Euler's Equations* (II) (in Russian). Dokl. Akad. Nauk, U.S.S.R., **202** (1967).

ECUAȚIILE LUI LAGRANGE PENTRU UN RIGID ȘI INTERPRETĂRI GEOMETRICE PRIVIND PRINCIPIILE IMPULSULUI ȘI MOMENTULUI CINETIC

(Rezumat)

Se consideră varietatea riemanniană cu șase dimensiuni a variabilelor $(x_{1C}, x_{2C}, x_{3C}; \theta^1, \theta^2, \theta^3)$ — parametrii de poziție ai rigidului, înzestrată cu metrica $ds^2 = \mathcal{E}dt$, reprezentând energia cinetică a rigidului. Se calculează geodezicele acestei structuri subliniind legătura cu ecuațiile lui Lagrange pentru solidul rigid.

Se arată că cele șase ecuații ale lui Lagrange pentru rigid coincid cu ecuațiile rezultând din principiul impulsului, respectiv principiul momentului cinetic pentru solidul rigid. Ultimele trei ecuații corespundătoare principiului momentului cinetic, deduse din ecuațiile lui Lagrange, aparțin unei structuri geometrice care nu este nici măcar local euclidiană. Calculând coeficienții de conexiune ai acestei structuri se scot în evidență interpretările diferite ce se pot da acelorăși mărimi tensoriale în cele două structuri (transport paralel diferit, etc.).



A GENERALIZED DYNAMICAL THEORY OF THERMOELASTICITY WITH MICROSTRUCTURE

BY

D. VIERU and SILVIA G. CIUMAȘU

1. Introduction. In this paper we propose an alternate formulation, [4], of the generalized non-linear thermoelasticity solid problem with microstructure. Our approach is motivated by the works [3], [5]. In contrast to the conventional theories based on parabolic-type heat equation, this theory involve a hyperbolic-type heat equation. We deduce, along with thermodynamical restriction on the material constants, the governing equations of the linearized theory for anisotropic and isotropic solid.

2. The Balance-Equations of Non-Linear Theory. We consider an arbitrary material volume $\mathcal{O}$ in the continuum bounded by a surface $\partial\mathcal{O}$ at time t , and we suppose that V is the corresponding volume in the initial undeformed state of the continuum, bounded by a surface ∂V , and refer motion to a fixed system of rectangular cartesian axes Ox_i ($i=1, 2, 3$).

The motion of a generical particle of body, at time, t , is given by

$$(2.1) \quad x_i = x_i(X_A, t), \quad \varphi_{iA} = \varphi_{iA}(X_B, t), \quad (X_A, t) \in \bar{D} \times [0, t_1].$$

We suppose that

$$(2.2) \quad \det(x_{i,A}) > 0, \quad \det(\varphi_{iA}) > 0;$$

$$x_i(X_A, 0) = X_A, \quad \varphi_{iA}(x_B, t) = \delta_{iA}; \quad x_i, \varphi_{iA} \in C^2[\bar{D} \times [0, t_1]].$$

We start with the following fundamental equations:

i) *Strain-displacement relations*

$$(2.3) \quad \begin{aligned} u_i &= x_i - X_A \delta_{iA} = x_i - X_i, \quad u_{iA} = \varphi_{iA} - \delta_{iA}, \\ 2E_{AB} &= x_{i,A} x_{i,B} - \delta_{AB} = u_{A,B} + u_{B,A} + u_{M,A} u_{M,B}, \\ F_{AB} &= x_{i,A} \varphi_{iB} - \delta_{AB} = u_{AB} + u_{M,A} u_{MB} + u_{M,A} \delta_{MB}, \\ S_{AB} &= \frac{1}{2} (F_{AB} + F_{BA}), \quad R_{AB} = \frac{1}{2} (F_{AB} - F_{BA}), \end{aligned}$$

where

$$u_A = u_i \delta_{iA}, \quad u_{AB} = u_{iA} \delta_{iB}.$$

ii) *The balance equations*

mass balance

$$(2.4) \quad \frac{d}{dt} \int_V \rho \, dv = 0;$$

momentum balance

$$(2.5) \quad \int_V \rho_0 \dot{V}_i dv = \int_V \rho_0 F_i dv + \int_{\partial V} T_i dA;$$

spin generalized balance

$$(2.6) \quad \int_V \rho_0 I_{AB} \dot{V}_{iB} dv = \int_V (\rho_0 N_{iA} + M_{iA}) dv;$$

momentum of momentum balance

$$(2.7) \quad \int_V \rho_0 e_{ijk} (x_j \dot{V}_k + I_{AB} \varphi_{jA} \dot{V}_{kB}) dv = \int_{\partial V} e_{ijk} x_j T_k dA + \int_V \rho_0 e_{ijk} (x_j F_k + \varphi_{jA} N_{kA}) dv;$$

energy balance

$$(2.8) \quad \int_V \rho_0 (V_i \dot{V}_i + I_{AB} V_{iA} \dot{V}_{iA} + \dot{U}) dv = \\ = \int_{\partial V} (T_i V_i - Q) dA + \int_V \rho_0 (F_i V_i + V_{iA} N_{iA} + r) dv;$$

Green's laws inequality

$$(2.9) \quad \int_V \rho_0 \dot{S} dv - \int_V \frac{\rho_0 r}{\Phi} dv + \int_{\partial V} \frac{Q}{\Phi} dA \geq 0.$$

In the material description from (2.4) – (2.9) the balance local equations take the form:

$$(2.10) \quad \rho_0 = J \rho,$$

$$(2.11) \quad T_{Ai,A} + \rho_0 F_i = \rho_0 \dot{V}_i,$$

$$(2.12) \quad \rho_0 N_{iA} + M_{iA} = \rho_0 I_{AB} \dot{V}_{iB},$$

$$(2.13) \quad \rho_0 \dot{U} = \rho_0 r - \dot{E}_{AB} \bar{S}_{AB} + \dot{F}_{AB} M_{AB} - Q_{A,A},$$

$$(2.14) \quad \rho_0 \Phi \dot{S} - \rho_0 r + Q_{A,A} - \frac{Q_A \Phi_{,A}}{\Phi} \geq 0,$$

$$(2.15) \quad T_{Ai} N_A = T_i, \quad Q_A N_A = Q.$$

The following notations are used: ρ – density, $J = \det(x_{k,A})$, $V_k = \dot{u}_k$ – velocity, $V_{iB} = \dot{\phi}_{iB}$ – dipolar velocity, F_i – external body force per unit mass, N_{iA} – external body couple per unit mass, T_{Ai} – first Piola-Kirchhoff stress tensor, T_{AB} – second Piola-Kirchhoff stress tensor, M_{iA} – internal body couple per unit volume, $M_{AB} = X_{A,i} M_{iB}$, $\bar{S}_{AB} = T_{AB} - \varphi_{iC} X_{B,i} M_{AC}$, Q_K – heat flux vector per unit surface in the reference configuration, r – internal energy supply per unit mass, U – internal energy per unit mass, S – entropy function per unit mass, $\Phi (> 0)$ – scalar function.

Elimination of r between Eqs. (2.13) – (2.14) and introduction of a scalar function $\psi = U - S\Phi$, yields

$$(2.16) \quad -\rho_0 (\dot{\psi} + S\dot{\Phi}) + \dot{E}_{AB} \bar{S}_{AB} + \dot{F}_{AB} M_{AB} - \frac{Q_A \Phi_{,A}}{\Phi} \geq 0.$$

We consider the following constitutive variables : $X = (E_{AB}, F_{AB}, \theta, \theta_{,A}, \dot{\theta})$ — where θ is an empirical temperature — and the functionals constitutive $Y = (\psi, \Phi, S, \bar{S}_{AB}, M_{AB}, Q_A)$. As in [3] we can provide that the following relations were satisfied

$$(2.17) \quad \bar{S}_{AB} = \frac{\partial \sigma}{\partial E_{AB}}, \quad M_{AB} = \frac{\partial \sigma}{\partial F_{AB}}, \quad \text{or} \quad M'_{AB} = \frac{\partial \sigma}{\partial S_{AB}}, \quad M''_{AB} = \frac{\partial \sigma}{\partial R_{AB}},$$

$$(2.18) \quad Q_A = -\Phi \frac{\partial \sigma}{\partial \theta_{,A}} \bigg/ \frac{\partial \Phi}{\partial \dot{\theta}}, \quad \rho_0 S = -\frac{\partial \sigma}{\partial \dot{\theta}} \bigg/ \frac{\partial \Phi}{\partial \dot{\theta}},$$

$$(2.19) \quad \left(\frac{\partial \sigma}{\partial \theta} + \rho_0 S \frac{\partial \Phi}{\partial \dot{\theta}} \right) \dot{\theta} + \frac{Q_k}{\Phi} \frac{\partial \Phi}{\partial \theta} \theta_{,k} \leq 0,$$

$$(2.20) \quad \Phi = \Phi(\theta, \dot{\theta}), \quad Q_A \neq 0, \quad \Phi(\theta, 0) = \theta_0 + \theta,$$

$$(2.21) \quad \rho_0 \dot{S} \Phi + \left(\frac{\partial \sigma}{\partial \theta} + \rho_0 S \frac{\partial \Phi}{\partial \dot{\theta}} \right) \dot{\theta} + \frac{\partial \sigma}{\partial \theta_{,A}} \dot{\theta}_{,A} - \rho_0 r + Q_{A,A} = 0,$$

where

$$\sigma = \rho_0 \psi, \quad \sigma = \sigma(E_{AB}, F_{AB}, \theta, \theta_{,A}, \dot{\theta}).$$

Together with kinematic Eqs. (2.3), balance Eqs. (2.11) and (2.21), these equations form the full set of fundamental equations of the theory.

3. Linearized Theory Equations. We now deduce from equations of Sec. 2, the field equations of the linearized theory. We suppose that the temperature θ is measured from a constant reference temperature θ_0 and

$$(3.1) \quad |u_i| \ll 1, \quad |u_{i,A}| \ll 1, \quad |u_{iA}| \ll 1, \quad (X_{A,i}) \in \bar{D} \times [0, t_1].$$

The strain displacement tensor [5] reduces to the infinitesimal strain tensor

$$(3.2) \quad \begin{aligned} E_{AB} &\simeq \frac{1}{2}(u_{A,B} + u_{B,A}) = \varepsilon_{AB}, & F_{AB} &= u_{B,A} + u_{AB}, \\ S_{AB} &\simeq \frac{1}{2}(u_{A,B} + u_{B,A}) + \frac{1}{2}(u_{AB} + u_{BA}) = \varepsilon_{AB} + e_{AB}, \\ R_{AB} &\simeq \frac{1}{2}(u_{AB} - u_{BA}) - \frac{1}{2}(u_{A,B} - u_{B,A}) = r_{AB}, \\ e_{AB} &= \frac{1}{2}(u_{AB} + u_{BA}), & r_{AP} &= \Omega_{AB} - \omega_{AB}, \\ \Omega_{AB} &= \frac{1}{2}(u_{AB} - u_{BA}), & \omega_{AB} &= \frac{1}{2}(u_{A,B} - u_{B,A}). \end{aligned}$$

For the linearized theory of an anisotropic solid, assuming that the initial body is free from stress and has zero heat flux, we obtain the constitutive equations:

$$(3.3) \quad \sigma_{ij} = \frac{\partial \sigma}{\partial \varepsilon_{ij}} = E_{ij}^*(\theta + \alpha \dot{\theta}) + B_{ijkl}^* \varepsilon_{kl} + C_{kl ij}^* e_{kl} + D_{kl ij}^* r_{kl},$$

$$(3.4) \quad \tau_{ij} = \frac{\partial \sigma}{\partial r_{ij}} = E_{ij}''(\theta + \alpha \dot{\theta}) + D_{ijkl}^* \varepsilon_{kl} + C'_{ijkl} e_{kl} + B''_{ijkl} r_{kl},$$

$$(3.5) \quad \mu_{ij} = \frac{\partial \sigma}{\partial e_{ij}} = E'_{ij}(\theta + \alpha \dot{\theta}) + C^*_{klij} \varepsilon_{kl} + B'_{ijkl} e_{kl} + C'_{ijkl} r_{kl},$$

$$(3.6) \quad t_{ij} = \sigma_{ij} + \tau_{ij}, \quad m_{ij} = \mu_{ij} + \tau_{ij},$$

$$(3.7) \quad q_i = -\theta_0(b_i \dot{\theta} + k_{ij} \theta_{,j}),$$

$$(3.8) \quad \rho_0 S = a + d\theta + h\dot{\theta} - b_i \theta_{,i} - E^*_{ij} \varepsilon_{ij} - E'_{ij} e_{ij} - E''_{ij} r_{ij},$$

$$(3.9) \quad \begin{aligned} \sigma &= \sigma_0 - a\theta - b\dot{\theta} - \frac{1}{2} d\theta^2 - e\theta\dot{\theta} - \frac{1}{2} f\dot{\theta}^2 + \alpha b_i \dot{\theta} \theta_{,i} + \frac{1}{2} \alpha k_{ij} \theta_{,i} \theta_{,j} + \\ &+ E^*_{ij} \varepsilon_{ij}(\theta + \alpha \dot{\theta}) + E'_{ij} e_{ij}(\theta + \alpha \dot{\theta}) + E''_{ij} r_{ij}(\theta + \alpha \dot{\theta}) + \frac{1}{2} B^*_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \\ &+ C^*_{ijkl} \varepsilon_{ij} e_{kl} + D^*_{ijkl} r_{ij} \varepsilon_{kl} + \frac{1}{2} B'_{ijkl} r_{ij} r_{kl} + C'_{ijkl} r_{ij} e_{kl} + \frac{1}{2} B'_{ijkl} e_{ij} e_{kl}, \\ \Phi &= \theta_0 + \theta + \alpha \dot{\theta} + \beta \theta \dot{\theta} + \frac{1}{2} \gamma \dot{\theta}^2. \end{aligned}$$

Motion equations are

$$(3.10) \quad \rho \ddot{u}_i = \sigma_{i,k,k} + \tau_{i,k,k} + f_i,$$

$$(3.11) \quad \rho I \ddot{\varphi}_{ij} = \mu_{ij} + \tau_{ij} + f_{ij},$$

where

$$f_i = \rho F_i, \quad f_{ij} = \rho N_{ij}, \quad \rho = \rho_0 \text{ in } V.$$

The boundary conditions are

$$(3.12) \quad t_{kl} n_l = \sigma_{kl} n_l - \tau_{kl} n_l = T_k \quad \text{on } \partial V.$$

Energy equation can be written in the form

$$(3.13) \quad \rho \theta_0 \dot{S} + q_{i,i} = \rho r.$$

The coefficients in (3.3) – (3.9) have the following symmetries

$$(3.14) \quad \begin{aligned} B^*_{ijkl} &= B^*_{klij} = B^*_{jikl}, & D^*_{ijkl} &= -D^*_{jikl} = D^*_{ijlk}, \\ B''_{ijkl} &= -B''_{jikl} = -B''_{ijlk}, & C^*_{ijkl} &= C^*_{jikl} = C^*_{ijlk}, \\ B'_{ijkl} &= B'_{jikl} = B'_{ijlk}, & C'_{ijkl} &= C'_{jikl} = -C'_{ijlk}. \end{aligned}$$

For an isotropic solid we have

$$(3.15) \quad \begin{aligned} \sigma &= \frac{1}{2} \tilde{\lambda} (\text{tr} \varepsilon)^2 + \tilde{\mu} \text{tr} \varepsilon^2 + r (\text{tr} \varepsilon) (\text{tr} \varepsilon) + 2 \text{str}(\varepsilon \varepsilon) + \frac{1}{2} p (\text{tr} \varepsilon)^2 + q \text{tr} \varepsilon^2 - \\ &- \frac{1}{2} C \text{tr} \varepsilon^2 + (A + B) (\text{tr} \varepsilon) (\theta + \alpha \dot{\theta}) + B (\text{tr} \varepsilon) (\theta + \alpha \dot{\theta}), \end{aligned}$$

$$(3.16) \quad \sigma_{kl} = \tilde{\lambda} \varepsilon_{mm} \delta_{kl} + 2 \tilde{\mu} \varepsilon_{kl} + r e_{mm} \delta_{kl} + 2 s e_{kl} + (A + B) (\theta + \alpha \dot{\theta}) \delta_{kl} = \frac{1}{2} (t_{kl} + t_{lk}),$$

$$(3.17) \quad \tau_{kl} = -c r_{kl} = \frac{1}{2} (t_{kl} - t_{lk}),$$

$$(3.18) \quad \mu_{kl} = r \varepsilon_{mm} \delta_{kl} + 2 s \varepsilon_{kl} + p e_{mm} \delta_{kl} + 2 q e_{kl} + B (\theta + \alpha \dot{\theta}) \delta_{kl},$$

$$(3.19) \quad q_i = -\theta_0 k \theta_{,i},$$

$$(3.20) \quad \rho S = a + d\theta + h\dot{\theta} - (A + B) \varepsilon_{ii} - B e_{ii},$$

where $\tilde{\lambda}$, $\tilde{\mu}$, r , s , p , q , c [5], A , B , are material constants for an isotropic body.

The above Eqs. (3.3) — (3.13), (3.10) — (3.13) and (3.15) — (3.20) generalize the equations considered in [5] for thermoelasticity of solids with microstructure.

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REFERENCES

1. Boschi E., Ieșan D., *A Generalized Theory of Linear Micropolar Thermoelasticity*. Meccanica, VIII, 3 (1977).
2. Green F., Laws N., *On the Entropy Production Inequality*. Arch. Rat. Mech. Anal., 45, 47—59 (1972).
3. Green E., Lindsay K. A., *Thermoelasticity*. J. Elasticity, 2, 1—7 (1972).
4. Neagu S. G., *A Generalized Theory of Microstructure in Linear Thermoelasticity*. Bul. Inst. Polit. Iași, XXVI(XXX), 3—4, s. 1 (1980).
5. Soós E., *Modele discrete și continue ale solidelor*. Ed. șt., Buc., 1974.

O TEORIE DINAMICĂ GENERALIZATĂ A TERMOELASTICITĂȚII CU MICROSTRUCTURĂ

(Rezumat)

Se propune o altă formulare a teoriei termoelasticității generalizate cu microstructură pe baza lucrărilor [5], [3]. Se prezintă ecuațiile de bază pentru teoria neliniară și pentru teoria liniarizată în cazul mediilor anizotrope și izotrope.

LARGE AND INFINITELY SMALL ROTATIONS IN $\mathbb{R}^3$

BY

ALFRED BRAIER and SORIN NEAGU

1. The peculiarities of large rotations of rigid bodies [1] ... [6], well explored at the level of kinematic modeling, and the deep involvement of the infinitely small rotations at the dynamic level, arouses again interest in the present context of robotic mechanisms or nonlinear finite element analysis [7].

The present paper presents in a simple and direct way the most significant results regarding the large and infinitely small rigid rotations in $\mathbb{R}^3$, underlining the existing connections between their characteristic parameters.

2. Let be

$$(1) \quad \hat{X} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \hat{X}' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}; \quad \hat{V} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix}, \quad \hat{V}' = \begin{pmatrix} v_{x'} \\ v_{y'} \\ v_{z'} \end{pmatrix}$$

the column matrices attached to the position vector and velocity of a point in the rigid body, written in the fixed frame (S) and respectively in the mobile frame (S'), that is to $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}'$ and $\mathbf{v} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} = v_{x'}\mathbf{i}' + v_{y'}\mathbf{j}' + v_{z'}\mathbf{k}'$. Also

$$(2) \quad \mathcal{R} = \begin{pmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} \end{pmatrix}, \quad \tilde{\omega} = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix}, \quad \tilde{\omega}' = \begin{pmatrix} 0 & -\omega_{z'} & \omega_{y'} \\ \omega_{z'} & 0 & -\omega_{x'} \\ -\omega_{y'} & \omega_{x'} & 0 \end{pmatrix}$$

are the orthogonal rotation matrix (with $\det \mathcal{R} = 1$ and obviously $\mathcal{R}^T = \mathcal{R}^{-1}$) as well as the skew-symmetric matrices attached to the angular velocity vector in the frames (S) and (S').

With the routine equations

$$(3) \quad \dot{\hat{X}} = \mathcal{R}\dot{\hat{X}}', \quad \dot{\hat{V}} = \dot{\mathcal{R}}\hat{X}' + \mathcal{R}\dot{\hat{V}}' = \dot{\mathcal{R}}\hat{X}' + \mathcal{R}\tilde{\omega}'\hat{X}',$$

where dots indicate d/dt , and taking into account that $\dot{\hat{X}}' = \mathcal{R}^T\dot{\hat{X}}$, $\dot{\hat{V}}' = \tilde{\omega}'\hat{X}'$, we get

$$(4) \quad \tilde{\omega} = \dot{\mathcal{R}}\mathcal{R}^T (= -\mathcal{R}\dot{\mathcal{R}}^T), \quad \tilde{\omega}' = \mathcal{R}^T\tilde{\omega} (= -\mathcal{R}^T\dot{\mathcal{R}}).$$

The expressions in brackets result by differentiating the equalities $\mathcal{R}\mathcal{R}^T = \mathcal{R}^T\mathcal{R} = I_3$, I_3 being the unity matrix.

Let C_0 , C and C_1 denote the rigid body positions at the moments t_0 , t and $t_1 = t + dt$. The large rotation $C_0 \rightarrow C$ is described by the orthogonal matrix $\mathcal{R}$, the large rotation $C_0 \rightarrow C_1$ by the same type matrix $\mathcal{R}_1 = \mathcal{R} + d\mathcal{R}$, while the infinitely small rotation $C \rightarrow C_1$ is described by the skew-symmetric matrix attached to the angular velocity vector w . Equations (4) reflect the fact that the infinitely small displacement (rotation about a fixed point) $C \rightarrow C_1$ is equivalent to the „difference“ of the two (large) displacements (rotations) $C_0 \rightarrow C_1$ and $C_0 \rightarrow C$.

The previous remarks, particularly (4), allow to calculate in a simple manner the relations between the angular velocity, measuring the rigid body instantaneous rotation at a given time t , and the elements of the large rotation carrying the body from a previous well defined position to that at this very moment t ; such elements are present in the rotation matrix (see also [1], [5]).

For instance, bearing in mind (4) we get

$$\begin{aligned} \omega_x &= \dot{\alpha}_{31}\alpha_{21} + \dot{\alpha}_{32}\alpha_{22} + \dot{\alpha}_{33}\alpha_{23} = -(\dot{\alpha}_{21}\alpha_{31} + \dot{\alpha}_{22}\alpha_{32} + \dot{\alpha}_{23}\alpha_{33}), \\ (5) \quad \omega_y &= \dot{\alpha}_{11}\alpha_{31} + \dot{\alpha}_{12}\alpha_{32} + \dot{\alpha}_{13}\alpha_{33} = -(\dot{\alpha}_{31}\alpha_{11} + \dot{\alpha}_{32}\alpha_{12} + \dot{\alpha}_{33}\alpha_{13}), \\ \omega_z &= \dot{\alpha}_{21}\alpha_{11} + \dot{\alpha}_{22}\alpha_{12} + \dot{\alpha}_{23}\alpha_{13} = -(\dot{\alpha}_{11}\alpha_{21} + \dot{\alpha}_{12}\alpha_{22} + \dot{\alpha}_{13}\alpha_{23}) \end{aligned}$$

and also

$$\begin{aligned} \omega_{x'} &= \dot{\alpha}_{12}\alpha_{13} + \dot{\alpha}_{22}\alpha_{23} + \dot{\alpha}_{32}\alpha_{33} = -(\dot{\alpha}_{13}\alpha_{12} + \dot{\alpha}_{23}\alpha_{22} + \dot{\alpha}_{33}\alpha_{32}), \\ (6) \quad \omega_{y'} &= \dot{\alpha}_{13}\alpha_{11} + \dot{\alpha}_{23}\alpha_{21} + \dot{\alpha}_{33}\alpha_{31} = -(\dot{\alpha}_{11}\alpha_{13} + \dot{\alpha}_{21}\alpha_{23} + \dot{\alpha}_{31}\alpha_{33}), \\ \omega_{z'} &= \dot{\alpha}_{11}\alpha_{12} + \dot{\alpha}_{21}\alpha_{22} + \dot{\alpha}_{31}\alpha_{32} = -(\dot{\alpha}_{12}\alpha_{11} + \dot{\alpha}_{22}\alpha_{21} + \dot{\alpha}_{32}\alpha_{31}). \end{aligned}$$

3. As we are aware, the matrices for rotations through angles α , β , γ about the cartesian axes Ox , Oy , Oz respectively are

$$(7) \quad \mathcal{R}_x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \alpha & -\sin \alpha \\ 0 & \sin \alpha & \cos \alpha \end{pmatrix}, \quad \mathcal{R}_y = \begin{pmatrix} \cos \beta & 0 & \sin \beta \\ 0 & 1 & 0 \\ -\sin \beta & 0 & \cos \beta \end{pmatrix}, \quad \mathcal{R}_z = \begin{pmatrix} \cos \gamma & -\sin \gamma & 0 \\ \sin \gamma & \cos \gamma & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Applying, for instance (4₁), we find

$$(8) \quad \omega_x = \dot{\mathcal{R}}_x \mathcal{R}_x^T = \dot{\alpha} \tilde{I}_x, \quad \omega_y = \dot{\mathcal{R}}_y \mathcal{R}_y^T = \dot{\beta} \tilde{I}_y, \quad \omega_z = \dot{\mathcal{R}}_z \mathcal{R}_z^T = \dot{\gamma} \tilde{I}_z,$$

where

$$(9) \quad \tilde{I}_x = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \tilde{I}_y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \tilde{I}_z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are the three linear independent generators of the Lie group of three dimensional rotations $SO(3)$ [8].

The sequence of rotations about the three mentioned axes, the order being for instance $Ox - Oy - Oz$, determines an overall transformation matrix $\mathcal{R} = \mathcal{R}_z \mathcal{R}_y \mathcal{R}_x$. With (4₁) we have

$$\tilde{\omega} = (\dot{\mathcal{R}}_z \mathcal{R}_y \mathcal{R}_x + \mathcal{R}_z \dot{\mathcal{R}}_y \mathcal{R}_x + \mathcal{R}_z \mathcal{R}_y \dot{\mathcal{R}}_x) \mathcal{R}_x^T \mathcal{R}_y^T \mathcal{R}_z^T$$

and according to (7) it becomes

$$(10) \quad \tilde{\omega} = \gamma \tilde{I}_z + \beta \tilde{I}_y \mathcal{R}_z^T + \alpha \mathcal{R}_z \mathcal{R}_y \tilde{I}_x \mathcal{R}_y^T \mathcal{R}_z^T.$$

This equation shows clearly that the infinitely small rotation expressed by $\tilde{\omega} \leftrightarrow \mathbf{w}$ is the superposition of three infinitely small rotations measured by α , β and γ , about the directions denoted by the unit vectors

$$(11) \quad \begin{aligned} \mathcal{R}_z \mathcal{R}_y \tilde{I}_x \mathcal{R}_y^T \mathcal{R}_z^T &\leftrightarrow \mathbf{i}_\alpha = \cos \beta \cos \gamma \mathbf{i} + \cos \beta \sin \gamma \mathbf{j} - \sin \beta \mathbf{k}, \\ \mathcal{R}_z \tilde{I}_y \mathcal{R}_z^T &\leftrightarrow \mathbf{i}_\beta = -\sin \gamma \mathbf{i} + \cos \gamma \mathbf{j}, \\ \tilde{I}_z &\leftrightarrow \mathbf{i}_\gamma = \mathbf{k}. \end{aligned}$$

In the mobile frame (S'), with (4₂) we find

$$(12) \quad \tilde{\omega}' = \alpha \tilde{I}'_x + \beta \mathcal{R}_x^T \tilde{I}'_y \mathcal{R}_x + \gamma \mathcal{R}_x^T \mathcal{R}_y^T \tilde{I}'_z \mathcal{R}_y \mathcal{R}_x,$$

where the matrices $\tilde{I}'_x$, $\tilde{I}'_y$, $\tilde{I}'_z$ are given by the same relations (9), but „readed“ in the frame (S'). The unit vectors, analogous to (11), are now

$$(13) \quad \begin{aligned} \tilde{I}'_x &\leftrightarrow \mathbf{i}'_\alpha = \mathbf{i}', \\ \mathcal{R}_x^T \tilde{I}'_y \mathcal{R}_x &\leftrightarrow \mathbf{i}'_\beta = \cos \alpha \mathbf{j}' - \sin \alpha \mathbf{k}', \\ \mathcal{R}_x^T \mathcal{R}_y^T \tilde{I}'_z \mathcal{R}_y \mathcal{R}_x &\leftrightarrow \mathbf{i}'_\gamma = -\sin \beta \mathbf{i}' + \sin \alpha \cos \beta \mathbf{j}' + \cos \alpha \cos \beta \mathbf{k}'. \end{aligned}$$

Without difficulty we can write the components on the axes of frames (S) and respectively (S'), of the angular velocity $\mathbf{w}$ just obtained.

In a perfectly similar way results the customary relation between the angular velocity and Euler angles.

4. Let consider now a large rotation through angle θ about the direction fixed by the unit vector $\mathbf{e} = e_1 \mathbf{i} + e_2 \mathbf{j} + e_3 \mathbf{k}$ ($= e'_1 \mathbf{i}' + e'_2 \mathbf{j}' + e'_3 \mathbf{k}'$). In [6] Argyris obtained a very concise and elegant form for the rotation matrix, namely

$$(14) \quad \mathcal{R} = I_3 + \sin \theta \tilde{e} + (1 - \cos \theta) \tilde{e}^2,$$

where I_3 is the unit matrix and $\tilde{e}$ — the unsymmetric matrix attached to the unit vector $\mathbf{e}$.

Writting (14) as

$$(15) \quad \mathcal{R} = I_3 + 2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} \tilde{e} + 2 \sin^2 \frac{\theta}{2} \tilde{e}^2,$$

we express immediately the elements of $\mathcal{R}$ through the Euler-Rodrigues parameters [3]

$$(16) \quad \lambda_0 = \cos \frac{\theta}{2}, \quad \lambda_1 = \sin \frac{\theta}{2} e_1, \quad \lambda_2 = \sin \frac{\theta}{2} e_2, \quad \lambda_3 = \sin \frac{\theta}{2} e_3$$

($\lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 = 1$), thus obtaining

$$\begin{aligned}
 \alpha_{11} &= \cos \theta + (1 - \cos \theta) e_1^2 = \lambda_0^2 + \lambda_1^2 - \lambda_2^2 - \lambda_3^2, \\
 \alpha_{12} &= -\sin \theta e_3 + (1 - \cos \theta) e_1 e_2 = 2(\lambda_1 \lambda_2 - \lambda_0 \lambda_3), \\
 \alpha_{13} &= \sin \theta e_2 + (1 - \cos \theta) e_3 e_1 = 2(\lambda_1 \lambda_3 + \lambda_0 \lambda_2), \\
 (17) \quad \alpha_{21} &= \sin \theta e_3 + (1 - \cos \theta) e_1 e_2 = 2(\lambda_1 \lambda_2 + \lambda_0 \lambda_3), \\
 \alpha_{22} &= \cos \theta + (1 - \cos \theta) e_2^2 = \lambda_0^2 - \lambda_1^2 + \lambda_2^2 - \lambda_3^2, \\
 \alpha_{23} &= -\sin \theta e_1 + (1 - \cos \theta) e_2 e_3 = 2(\lambda_2 \lambda_3 - \lambda_0 \lambda_1), \\
 \alpha_{31} &= -\sin \theta e_2 + (1 - \cos \theta) e_3 e_1 = 2(\lambda_1 \lambda_3 - \lambda_0 \lambda_2), \\
 \alpha_{32} &= \sin \theta e_1 + (1 - \cos \theta) e_2 e_3 = 2(\lambda_2 \lambda_3 + \lambda_0 \lambda_1), \\
 \alpha_{33} &= \cos \theta + (1 - \cos \theta) e_3^2 = \lambda_0^2 - \lambda_1^2 - \lambda_2^2 + \lambda_3^2; \\
 &\left(\text{for instance, } \alpha_{11} = 1 - 2 \sin^2 \frac{\theta}{2} (e_2^2 + e_3^2) = \lambda_0^2 + \lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 2\lambda_2^2 - 2\lambda_3^2 \text{ etc.} \right).
 \end{aligned}$$

5. Let remind the quaternionic representation of large rotations [4]

$$(18) \quad q = \cos \frac{\theta}{2} + \check{e} \sin \frac{\theta}{2} = e^{\frac{\theta}{2} \check{e}} = \lambda_0 + \lambda_1 \check{i} + \lambda_2 \check{j} + \lambda_3 \check{k},$$

where q we call a „complete quaternion“, $\lambda_0 = \text{Re } q$ — the real part of q and $Qu q = \lambda_1 \check{i} + \lambda_2 \check{j} + \lambda_3 \check{k}$, a „purely quaternionic number“, its quaternionic part, attached here to the unit vector $\check{e}$, like $\check{i}$, $\check{j}$ and $\check{k}$ are attached to the unit vectors of system (S). The conjugate of q is

$$(19) \quad q^* = \cos \frac{\theta}{2} - \check{e} \sin \frac{\theta}{2} = e^{-\frac{\theta}{2} \check{e}} = \lambda_0 - \lambda_1 \check{i} - \lambda_2 \check{j} - \lambda_3 \check{k},$$

$$\text{with} \quad qq^* = q^*q = 1$$

(where obviously $\check{i}\check{j} = -\check{j}\check{i} = \check{k}$, $\check{j}\check{k} = -\check{k}\check{j} = \check{i}$, $\check{k}\check{i} = -\check{i}\check{k} = \check{j}$). We remind also that if a , b are two quaternions, then $ab = \text{Re}(ab) + Qu(ab)$ and the correspondences

$$(20) \quad \text{Re}(ab) \leftrightarrow -\mathbf{a} \cdot \mathbf{b}, \quad Qu(ab) \leftrightarrow \mathbf{a} \times \mathbf{b}$$

take place, where $\mathbf{a}$, $\mathbf{b}$ are the vectors in $\mathbb{R}^3$ represented by the quaternionic parts of a and b respectively.

In quaternionic language, equation (4₁) reads

$$(21) \quad \check{r} = q\check{r}'q^*$$

and (bearing in mind (22) and (20₂) the velocity is

$$\check{v} = \dot{q} \check{r}' q^* + \check{q} \check{r}' \dot{q}^* = \dot{q} \check{q}^* \check{q} \check{r}' q^* + \check{q} \check{r}' q^* \dot{q} \check{q}^* = \dot{q} \check{q}^* \check{r} - \check{r} \dot{q} \check{q}^*.$$

But

$$\dot{q} \check{q}^* \check{r} = \mathcal{R}e(\dot{q} \check{q}^* \check{r}) + Qu(\dot{q} \check{q}^* \check{r}), \quad \check{r} \dot{q} \check{q}^* = \mathcal{R}e(\check{q} \check{q}^* \check{r}) - Qu(\check{q} \check{q}^* \check{r})$$

and since $\mathbf{v} = \mathbf{w} \times \mathbf{r} \leftrightarrow \check{v} = Qu(\check{\omega} \check{r})$, we get that $\check{v} = 2 Qu(\dot{q} \check{q}^* \check{r})$, i.e.

$$(22) \quad \check{w} = 2 \dot{q} \check{q}^* (= -2 \check{q} \dot{q}^*); \quad \text{also} \quad \check{w}' = 2 \check{q}^* \dot{q} (= -2 \dot{q}^* \check{q}),$$

$\check{w}'$ being written in (S').

Using now the expressions (18₂) and (19₂) we arrive very easily to

$$(23) \quad \begin{aligned} \omega_x &= 2(\lambda_0 \dot{\lambda}_1 - \lambda_1 \dot{\lambda}_0 + \lambda_2 \dot{\lambda}_3 - \lambda_3 \dot{\lambda}_2), \\ \omega_y &= 2(\lambda_0 \dot{\lambda}_2 - \lambda_1 \dot{\lambda}_3 - \lambda_2 \dot{\lambda}_0 + \lambda_3 \dot{\lambda}_1), \\ \omega_z &= 2(\lambda_0 \dot{\lambda}_3 + \lambda_1 \dot{\lambda}_2 - \lambda_2 \dot{\lambda}_1 - \lambda_3 \dot{\lambda}_0); \\ \omega_{x'} &= 2(\lambda_0 \dot{\lambda}_1 - \lambda_1 \dot{\lambda}_0 - \lambda_2 \dot{\lambda}_3 + \lambda_3 \dot{\lambda}_2), \\ \omega_{y'} &= 2(\lambda_0 \dot{\lambda}_2 + \lambda_1 \dot{\lambda}_3 - \lambda_2 \dot{\lambda}_0 - \lambda_3 \dot{\lambda}_1), \\ \omega_{z'} &= 2(\lambda_0 \dot{\lambda}_3 - \lambda_1 \dot{\lambda}_2 + \lambda_2 \dot{\lambda}_1 - \lambda_3 \dot{\lambda}_0). \end{aligned}$$

Returning again to (18) and (19), (22₁) becomes first

$$2 \dot{q} \check{q}^* = \left[\dot{\theta} \left(-\sin \frac{\theta}{2} + \check{e} \cos \frac{\theta}{2} \right) + 2 \sin \frac{\theta}{2} \check{e} \right] \left(\cos \frac{\theta}{2} - \check{e} \sin \frac{\theta}{2} \right)$$

and since $\check{e}^2 = -1$, $\check{e} \check{e} = -\check{e} \check{e}$ we get

$$(24) \quad \check{w} = \dot{\theta} \check{e} + \sin \theta \check{e} + (1 - \cos \theta) \check{e} \check{e} \quad \text{or} \quad \mathbf{w} = \dot{\theta} \mathbf{e} + \sin \theta \mathbf{e} + (1 - \cos \theta) \mathbf{e} \times \mathbf{e},$$

in vectorial transcription.

Appealing to the (large) rotation vector

$$(25) \quad \mathbf{R} = \tan \frac{\theta}{2} \mathbf{e} \quad \text{with} \quad \dot{\mathbf{R}} = \frac{1}{2} \dot{\theta} \left(1 + \tan^2 \frac{\theta}{2} \right) \mathbf{e} + \tan \frac{\theta}{2} \dot{\mathbf{e}}$$

and eliminating $\dot{\mathbf{e}}$ from (24₂) we arrive to the tightly form [1]

$$(26) \quad \mathbf{w} = \frac{2}{1 + R^2} (\dot{\mathbf{R}} + \mathbf{R} \times \dot{\mathbf{R}}).$$

6. In terms of the Eulerian angles (φ — precession, ψ — proper rotation and χ — nutation), the quaternion (18) is [4]

$$(27) \quad q = e^{\frac{\varphi}{2} \check{k}} e^{\frac{\chi}{2} \check{j}} e^{\frac{\psi}{2} \check{k}},$$

leading to

$$(28) \quad \begin{aligned} \lambda_0 &= \cos \frac{\chi}{2} \cos \frac{\varphi + \psi}{2}, & \lambda_1 &= \sin \frac{\chi}{2} \cos \frac{\varphi - \psi}{2}, \\ \lambda_2 &= \sin \frac{\chi}{2} \sin \frac{\varphi - \psi}{2}, & \lambda_3 &= \cos \frac{\chi}{2} \sin \frac{\varphi + \psi}{2}. \end{aligned}$$

Reminding that if m, n are complex numbers, a spinor is

$$(29) \quad \mathbf{i} = m + nj, \quad j^2 = -1, \quad m, n \in \mathbb{C} \quad \text{and} \quad nj = jn^*;$$

any quaternion, in particular (18), can be written for instance as $q = (\lambda_0 + \lambda_3 \mathbf{k}) + (\lambda_2 - \lambda_1 \mathbf{k}) \mathbf{j}$. Then (with $\mathbf{i}$ and $\mathbf{j}$ instead of $\mathbf{k}$ and $\mathbf{j}$) it results

$$(30) \quad \begin{aligned} q &= (\lambda_0 + i\lambda_3) + (\lambda_2 - i\lambda_1) \mathbf{j}, & \omega &= i\omega_z + (\omega_y - i\omega_x) \mathbf{j}, \\ q^* &= (\lambda_0 - i\lambda_3) - (\lambda_2 - i\lambda_1) \mathbf{j}, \end{aligned}$$

From (28) we see that

$$(31) \quad \begin{aligned} \lambda_0 + i\lambda_3 &= \cos \frac{\chi}{2} e^{i\frac{\varphi + \psi}{2}} = \alpha, & -\lambda_2 + i\lambda_1 &= i \sin \frac{\chi}{2} e^{i\frac{\varphi - \psi}{2}} = \beta \end{aligned}$$

are (together with $\delta = \alpha^*$, $\gamma = -\beta^*$, $\alpha\delta - \beta\gamma = 1$) the famous Cayley-Klein parameters.

Then, the spinor attached to (30) and ultimately to a large rotation is

$$(32) \quad \mathbf{i} = \alpha - \beta \mathbf{j}; \quad \mathbf{i}^* = \delta + \beta \mathbf{j} \quad (\alpha^* = \delta, \quad \beta^* = -\gamma)$$

and (22) gives directly

$$(33) \quad \omega = 2i\mathbf{i}\dot{\mathbf{i}}^* (= -2\mathbf{i}\dot{\mathbf{i}}^*), \quad \omega' = 2\mathbf{i}^*\dot{\mathbf{i}} (= -2\mathbf{i}^*\dot{\mathbf{i}}).$$

Substituting (32) into (33) and comparing with (30₂) we obtain

$$\omega = 2i\mathbf{i}\dot{\mathbf{i}}^* = 2(\dot{\alpha} - \dot{\beta}\mathbf{j})(\delta + \beta\mathbf{j}) = 2[(\delta\dot{\alpha} - \gamma\dot{\beta}) + (\beta\dot{\alpha} - \alpha\dot{\beta})\mathbf{j}],$$

meaning that $\omega_z = 2\mathcal{J}m(\dot{\alpha}\dot{\alpha} - \beta\dot{\gamma})$, $\omega_y - i\omega_x = 2(\beta\dot{\alpha} - \alpha\dot{\beta})$.

Thus we get relations of direct interest in the dynamic of rigid bodies ([5] and especially [1]):

$$(34) \quad \begin{aligned} \omega_x + i\omega_y &= 2i(\beta\dot{\alpha} - \alpha\dot{\beta}), & \omega_{x'} + i\omega_{y'} &= 2i(\beta\dot{\delta} - \delta\dot{\beta}), \\ \omega_x - i\omega_y &= 2i(\gamma\dot{\delta} - \delta\dot{\gamma}), & \omega_{x'} - i\omega_{y'} &= 2i(\gamma\dot{\alpha} - \alpha\dot{\gamma}), \\ \omega_z &= 2i(\alpha\dot{\delta} - \beta\dot{\gamma}) = 2i(\gamma\dot{\beta} - \delta\dot{\alpha}); & \omega_{z'} &= 2i(\alpha\dot{\delta} - \gamma\dot{\beta}) = 2i(\beta\dot{\gamma} - \delta\dot{\alpha}); \end{aligned}$$

(for ω_z and $\omega_{z'}$ we used also the result of differentiating $\alpha\delta - \beta\gamma = 1$).

7. The parameters (31) are the elements of the matrix acting as unimodular operator in the two-dimensional complex space [8], assuring the Cayley-Klein parametrization of the $SU(2)$ group, namely

$$(35) \quad Q = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}, \quad Q^+ = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} = \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}, \quad QQ^+ = 1,$$

where Q^+ is the hermitian conjugate.

The hermitian matrices attached to the angular velocity vector in the frames (S) and (S') respectively being

$$(36) \quad W = \begin{pmatrix} \omega_z & \omega_x + i\omega_y \\ \omega_x - i\omega_y & -\omega_z \end{pmatrix}, \quad W' = \begin{pmatrix} \omega_{z'} & \omega_{x'} + i\omega_{y'} \\ \omega_{x'} - i\omega_{y'} & -\omega_{z'} \end{pmatrix}$$

the results (33) lead to

$$(37) \quad W = 2iQQ^+ (= -2i\dot{Q}Q^+), \quad W' = 2i\dot{Q}'Q' (= -2iQ'\dot{Q}').$$

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REFERENCES

1. Hamel G., *Theoretische Mechanik*. Springer Verlag, Berlin—Göttingen—Heidelberg, 1949.
2. Goldstein H., *Classical Mechanics*. Addison Wesley, Cambridge, 1956.
3. Synge J. L., *Classical Dynamics*. "Handbuch der Physik", vol. III/1, Springer Verlag, Berlin—Göttingen—Heidelberg, 1960.
4. Braier A., Enescu I., *Un opérateur quaternionique utilisé à l'étude des mouvements finis d'un système invariable*. Bul. Inst. Polit. Iași, **VI (X)**, 1—2, 61—84 (1960).
5. Лурье А. И., *Аналитическая механика*. Гос. изд. физ. мат. лит., Москва, 1961.
6. Argyris J., *An Excursion into Large Rotations*. Comput. Meth. in Appl. Mech. and Engrg., **32**, 1—3, 85—155 (1982).
7. Argyris et al., *Nonlinear Finite Element Analysis of Elastic Systems under Nonconservative Loading — Natural Formulation (I)*. Comput. Meths. in Appl. and Engrg., **26**, 75—125 (1981).
8. Teodorescu P. P., Nicorovici-Porumbaru N., *Aplicații ale teoriei grupurilor în mecanică și fizică*. Ed. tehn., Buc., 1985.

ROTAȚII FINITE ȘI INFINITEZIMALE ÎN $\mathbb{R}^3$

(Rezumat)

Se prezintă sub o formă simplă și directă rezultate semnificative privind rotațiile rigide, finite și infinitezimale, în $\mathbb{R}^3$. Subliniind conexiunile dintre mărimile ce le caracterizează, se scriu relațiile dintre vectorul viteză unghiulară și elementele rotațiilor finite, utilizând matricea de rotație (4), reprezentările cuaternionică (22), spinorială (33) și prin matricea operator unimodular în spațiul bi-dimensional complex (37).

AN EXPERIMENT FOR THE VERIFICATION OF THE EQUIVALENCE PRINCIPLE

BY

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1. Is it Necessary a New Eötvös-Braginski Type Experiment? A systematic study of the principle of equivalence and the theoretical interpretation of Eötvös experiments [1] has shown that the current generation of these experiments verifies the weak equivalence principle and the structure independence of m_p/m . By the very precise experiments performed in Princeton [2] and in Moscow [3], the principle of equivalence of inertial and gravitational mass for different materials, platinum, gold, and aluminium could be verified within stringent limits

$$(1) \quad \eta(\text{Au}, \text{Al}) = \left| \frac{(m_p/m)_{\text{Au}} - (m_p/m)_{\text{Al}}}{(m_p/m)_{\text{Au}}} \right| < 10^{-11} \text{ (Princeton experiment),}$$

$$(2) \quad \eta(\text{Pt}, \text{Al}) = \left| \frac{(m_p/m)_{\text{Pt}} - (m_p/m)_{\text{Al}}}{(m_p/m)_{\text{Pt}}} \right| < 10^{-12} \text{ (Moscow experiment).}$$

Furthermore, it has been shown that the gravity acceleration is independent of the composition of matter within one part in 10^{11} [4]. Thus it may be concluded that neutrons and protons bound in nuclei experience the same gravitational acceleration within about $10^{-10} \Delta g/g$. On the other hand the behaviour of free particles, atoms, neutrons, electrons, and photons in the earth's gravitational field has been studied with lower accuracy of about only one part in 10^2 or 10^3 .

If one analyses the free fall in an external static gravitational field of a composite test body consisting of electromagnetically interacting charged particles, one sees that Eötvös experiments accurate to better than a part in 10^{15} would be required to test the second-order parameter Λ_1 of Lightman and Lee which measures the deviation from a metric theory [5]. Thus, by improved Eötvös experiments, one may decide on the correctness of some theories of gravitation.

2. Comparison between Princeton and Moscow Experiments. It is well known that for a body of inertia I suspended from a torsion fibre, and having

a damping coefficient β and a proper period τ_0 , the Brownian motion occurs as oscillations of period τ_0 coherent over times of order $Q\tau_0$, where $Q = \pi/\beta\tau_0$ is the quality factor of the system [6]. The angular amplitude $\overline{\Delta\theta}$ of the oscillation is

$$(3) \quad \overline{\Delta\theta} \approx \frac{\tau_0}{2\pi} \sqrt{\frac{kT}{I}}.$$

In a classical Eötvös experiment, the masses of the torsion balance are simultaneously subject to gravity and to centrifugal acceleration from the Earth's rotation, and deviation from equivalence causes a torque

$$(4) \quad \Gamma^E = \frac{1}{2} \gamma m_b D f \sin \alpha,$$

where m_b is the sum of the masses, D — the length of the balance arm, f — the driving acceleration (the centrifugal acceleration of 1.4 cm/sec^2), and α — the angle between the balance arm and north. In order to avoid the elastic hysteresis in the suspension, Roll, Krotkov and Dicke [2] and Braginsky and Pano v [3] by keeping the apparatus fixed have used the sun as source, the driving acceleration being now only 0.6 cm/sec^2 .

The principal limitations on Princeton experiment were seismic disturbance, temperature effects and gravity gradient torques. Large shifts of the torsional modes of the balance were observed from construction activity near the instrument site; the experimenters were driven to working mostly at weekends.

If we consider a balance with unequal moments of inertia I_1, I_3 , situated at a distance R from a point mass M , the gravity gradient torque is [6]

$$(5) \quad \Gamma^g = \frac{3}{2} (I_3 - I_1) \frac{GM}{R^3} \sin 2\alpha,$$

where α is the angle of the principal axis of the balance to the vector $\mathbf{R}$. The ratio of amplitudes of the gravity gradient and Eötvös torques may be put in the form

$$(6) \quad \frac{\Gamma^g}{\Gamma^E} = \frac{3\pi J_2 D}{8 \gamma f} \frac{GM}{R^3},$$

where $I_3 - I_1 = J_2 m_b l^2$, m_b being the mass of the balance, l — the radius of gyration, and J_2 — the quadrupole coefficient for a balance with two masses made in half rings of diameter D . This equation may be transposed into a condition on the distance r which a body of density ρ and diameter d may be allowed to approach the balance before causing a disturbance comparable to the Eötvös torque

$$(7) \quad \frac{r}{d} > 0.85 \left(\frac{G J_2 D}{\gamma f} \right)^{1/3} \rho^{1/3}.$$

For Princeton apparatus with $J_2 D \approx 3 \times 10^{-2}$ and $\gamma f \approx 6 \times 10^{-12}$, r/d had to be less than $6.6\rho^{1/3}$. A 100 kg man could not approach closer than 4 m or a 10 ton truck closer than 18 m without disturbing the balance.

The Moscow apparatus was somewhat larger; it was made with light bodies rather than three to reduce gravity gradient torques. The period τ_0

was much larger — six hours instead of four minutes as in Princeton apparatus. The larger period increased the angular deflections for a given Eötvös signal, the predicted deflection for an η of 10^{-12} being 10^{-7} rad as compared with 2.4×10^{-10} rad in the Princeton apparatus. The Brownian amplitude, as calculated from Eq. (3), is 7.5×10^{-5} rad instead 4.3×10^{-7} rad. The natural damping time for Moscow balance ($I=400$ gm cm²) was about two years as compared with one year for Princeton apparatus ($I=270$ gm cm²). According to the paper [2] the natural Q of the Princeton apparatus was 10^5 and thus we may calculate the damping coefficient $\beta = \pi/\tau_0 Q$ which is not much different from Braginsky and Panov's.

3. The Jassy Gravitational Balance. The aim of the gravitational balance which is in construction at Jassy is twofold: first we will repeat an Eötvös experiment using the conclusions and the best achievements of the Princeton and Moscow experiment, and second we intend to have a versatile gravitational balance which can be utilised in various gravitational experiments.

The stainless steel vacuum chamber (Fig. 1) consists of three parts: balance can, stem (three meters long cylinder of 7 cm diameter) and hat which contains the mechanism for independently raising and lowering and rotating the torsion balance inside the vacuum chamber. Flanges are welded to the balance can and stem, while copper O rings provide the vacuum seal between the three parts.

The torsion pendulum is analogous to that of Braginsky and Panov but more precisely performed since it is not necessary to deform it when the balance is assembled. A Bayard-Alpert type ionization vacuum gauge is mounted on a glass-to-metal seal welded to the lower flange of the stem, and can be used to measure gas pressure in the vacuum chamber. A side arm from stainless steel stem, below to this gauge, is connected to the vacuum system (AV-500 aggregate and 1 000 litres oil diffusion pump with liquid nitrogen cold trap), which maintains vacuum inside the chamber of about 10^{-6} mm Hg. Furthermore, we intend to use a Vac-Ion pump during the experiment. For the backing procedure, the vacuum chamber is held at a temperature of about 300°C during a long time.

The instrumentation for measuring very small rotation of the torsion pendulum is shown schematically in Fig. 2. A He-Ne laser beam is reflected over a distance of about 10 m to a photographic film on a rotating drum, the drum being inclined to the light path at an angle of about 0.2 rad to increase the effective path length to 50 m. The spot is of about 0.3 cm wide. The attenuation of the laser beam is realized with a polarizer instead a grey filter as in Moscow experiment. Another novelty is the phototransistor triad which permits, due to a secondary beam, to damp the oscillation and to monitor the reflected beam on the photographic film. This is possible with aid of tension pulses applied to the electrodes which act on the aluminium cylinder fixed to suspension wire.

At present we are not sure that it is possible to improve the measurement of η but we will try. We take into account the criticism of the Braginsky and Panov's experiment as observed by Everitt [6]. According to the mirage formula the laser beam passing at distance l through a gas of refractive index n is deflected under a transverse temperature gradient dT/dx through a distance x_n ,

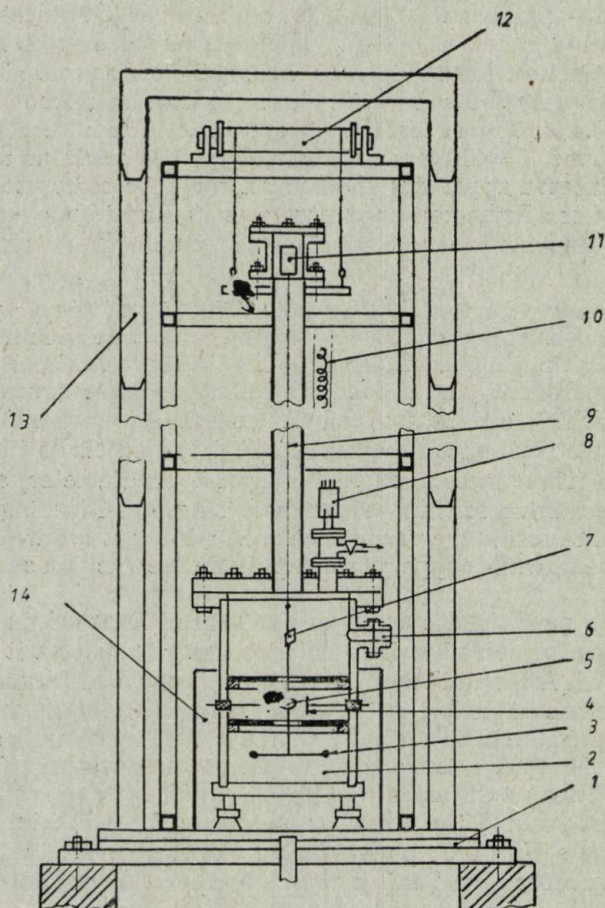


Fig. 1 — The torsion balance: 1 — turnover table; 2 — vacuum chamber; 3 — test bodies; 4 — capacitor; 5 — conductor of ellipsoidal form; 6 — optical window; 7 — mirror; 8 — vacuum ionization gauge; 9 — torsion fibre; 10 — degassing unit (350°C); 11 — device for adjustment in vacuum of a torsion pendulum; 12 — device for assembly; 13 — thermo-insulating material; 14 — magnetical screening box of permalloy.

(8)

$$x_n = \frac{n - l}{2n} \frac{l^2}{T} \frac{dT}{dX}.$$

If the light source stands next to the detector, x_n has to be doubled. With $l=12$ m, $x_n/l < 10^{-7}$ rad, and n for air at STP = 1.00029 the tranverse temperature gradient cannot exceed 7×10^{-5} °C/cm. This stability seems hard

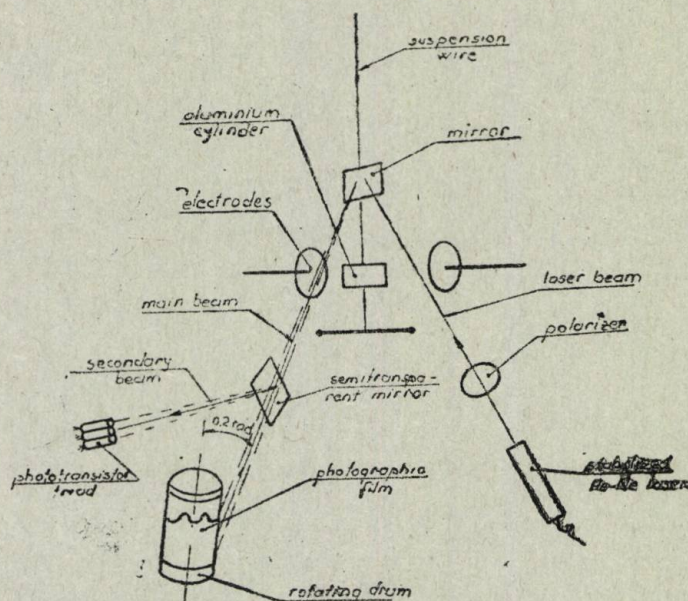


Fig. 2 — Device for measuring very small rotation of the torsion pendulum.

to achieve. To avoid this difficulty we intend to protect the laser beam by pipes with thermic insulation.

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REFERENCES

1. Haugan M. P., Will C. M., Phys. Rev., D **15**, 2 711 (1977).
2. Roll P. C., Kvotkov R., Dicke R. H., Ann. Phys. (N. Y.), **26**, 442 (1964).
3. Брагински В. Б., Панов В. И., ЖЕТФ, **61**, 875 (1971).
4. Koester L., Phys. Rev., D **14**, 907 (1976).
5. Lightman A. P., Lee D. L., Phys. Rev., **8**, 364 (1973).
6. Everitt C. W. F., in *Marcel Grossmann Meeting on General Relativity*. North-Holland Publ. Co., Amsterdam, 1977.

EXPERIMENT PENTRU VERIFICAREA PRINCIPIULUI ECHIVALENȚEI

(Rezumat)

Se prezintă proiectul și o parte din realizarea sa pentru studiul experimental al interacțiunilor gravitaționale în laborator. În principal, se urmărește verificarea principiului echivalenței maselor inerțiale și gravitaționale.

THE ANGULAR MOMENTUM IN CURVED SPACES

BY

S. OANCEA, GH. ZET and D. CONDURACHE

1. Introduction. In quantum mechanics the angular momentum is studied by means of the operators

$$\begin{aligned} \hat{L}_x &= -\frac{\hbar}{i} \left(\sin \varphi \frac{\partial}{\partial \theta} + \cos \varphi \cotan \theta \frac{\partial}{\partial \varphi} \right), \\ \hat{L}_y &= \frac{\hbar}{i} \left(\cos \varphi \frac{\partial}{\partial \theta} - \sin \varphi \cotan \theta \frac{\partial}{\partial \varphi} \right), \\ \hat{L}_z &= \frac{\hbar}{i} \frac{\partial}{\partial \varphi}, \end{aligned} \quad (1)$$

which are the Cartesian components of the angular momentum written in spherically coordinates and represent the elements of the matrix of the angular momentum tensor spatial part. In the special theory of relativity [1] this is given by

$$L_{\alpha\beta} = \begin{pmatrix} 0 & L_z & -L_y \\ -L_z & 0 & L_x \\ L_y & -L_x & 0 \end{pmatrix}. \quad (2)$$

2. Angular Momentum in Curved Spaces. In order to generalize the angular momentum (AM) for curved spaces, we shall study the alteration of the AM components while passing from the Minkowsky space to a spherically symmetric curved space. We shall also consider the functions and eigenvalues equation for a bosonic (scalar) field.

Let the spherically symmetric metric be [2]

$$ds^2 = R^2 dr^2 + Q(d\theta^2 + \sin^2 \theta d\varphi^2) - c^2 T^2 dt^2, \quad (3)$$

and the matrix of the covariant tetrad

$$a_{\alpha}^{\mu} = \begin{pmatrix} R \sin \theta \cos \varphi & Q \cos \theta \cos \varphi & -Q \sin \theta \sin \varphi & 0 \\ R \sin \theta \sin \varphi & Q \cos \theta \sin \varphi & Q \sin \theta \cos \varphi & 0 \\ R \cos \theta & -Q \sin \theta & 0 & 0 \\ 0 & 0 & 0 & cT \end{pmatrix}. \quad (4)$$

For a scalar field ψ we propose the function and the eigenvalues equation

$$(5) \quad \bar{L}_{ik}\psi = a_i^\alpha a_k^\beta L_{\alpha\beta}\psi,$$

where $\bar{L}_{ik}$ are the AM components in the curved space.

By using (2) and (4) in Eq. (5) we find

$$(6) \quad \begin{aligned} \bar{L}_{12} &= \frac{\hbar}{i} QR \frac{\partial}{\partial \theta}, \\ \bar{L}_{13} &= \frac{\hbar}{i} QR \frac{\partial}{\partial \varphi}, \\ \bar{L}_{23} &= 0. \end{aligned}$$

For the operator L^2 (second power of the AM) defined in [3] by the expression

$$(7) \quad L^2 = L^{ik} L_{ik} = L_{mn} g^{im} g^{kn} L_{ik},$$

the equation of functions and eigenvalues is

$$(8) \quad L^2\psi = -\hbar^2 \left(\frac{\partial^2}{\partial \theta^2} + \frac{1}{\sin^2 \theta} \frac{\partial^2}{\partial \varphi^2} \right) \psi,$$

which differs from that written in a flat space.

In order to determine the eigenvalues λ' of L^2 operator ($L^2\psi = \lambda'\psi$) we propose

$$\psi = \Phi(\varphi)\Theta(\theta).$$

The well known expression results for $\Phi(\varphi)$

$$\Phi = \Phi_0 e^{im\varphi}, \quad \text{where } m \text{ is integer.}$$

For Θ , the following equation is obtained

$$(9) \quad \frac{d^2\Theta}{d\theta^2} + \left(\lambda' - \frac{m^2}{\sin^2 \theta} \right) \Theta = 0.$$

We change the variable $\cos \theta = x$ and choose for Θ the expression

$$(10) \quad \Theta = (1-x^2)^{s/2} u(x),$$

where s is an integer and $u(x)$ a polynomial.

When substituting (10) in (9) one gets

$$(11) \quad (1-x^2)u'' - x(2s+1)u' + u \left(\lambda' - s^2 + \frac{s^2 - s - m^2}{1-x^2} \right) = 0.$$

In order to eliminate the singularity we have chosen $s^2 - s - m^2 = 0$, hence

$$(12) \quad m^2 = s(s-1).$$

Considering

$$(13) \quad u(x) = \sum a_k x^k,$$

the relation (11) leads to the recurrence formula

$$(k+2)(k+1)a_{k+2} = [k(k-1) + k(2s+1) - \lambda' + s^2]a_k.$$

But $a_{k+2}=0$ and $a_k \neq 0$, i.e.

$$(14) \quad \lambda' = s^2 + k(2s+1) - k(k-1).$$

Finally, for the AM values one obtains

$$(15) \quad L = \hbar \sqrt{s^2 + k(2s+1) - k(k-1)},$$

with $s(s-1)=m^2$ and s, m, k integers.

3. Conclusion. 1. Although the expression for the operator L^2 (8) is simple, the eigenvalues are quite elaborate.

2. The components of the AM depend on R and Q functions which give the gravitational field. Yet, L^2 does not depend on them.

3. The AM components separate the θ and φ variables and moreover the L_{23} component vanishes.

4. The study of the angular momentum in curved spaces can be developed by applying these operators to vectorial and tensorial fields.

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REFERENCES

1. Ougarov V., *Théorie de la Relativité Restreinte*, Ed. Mir, Moscou, 1979.
2. Mociuțchi C., Anal. Șt. Univ. Iași (s.n.), s. Ib., Fizică, **XX**, 2, 101 (1974).
3. Gottlieb I., Mociuțchi C., Oproiu V., Oancea S., Zet Gh., Anal. Șt. Univ. Iași (s. n.), s. Ib., Fizică, **XXVIII**, 33 (1982).

MOMENTUL UNGHIULAR ÎN SPAȚII CURBE

(Rezumat)

Se generalizează momentul unghiular pentru spații curbă, prin trecere de la spațiul Minkowsky la un spațiu curb cu simetrie sferică. Se aplică operatorii obținuți la cazul unui cîmp (scalar) bosonic.

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THE "HALL RESISTANCE" AND THE SPONTANEOUS SYMMETRY-BREAKING

BY

MARINA DARIESCU, MARCEL AGOP and C. DARIESCU

1. Introduction. One can obtain the quantification of the Hall resistance using an axial symmetric gauge invariant (U_1) Lagrangean with spontaneous symmetry-breaking.

2. The Axial Symmetric Gauge Invariant Lagrangean and the Field Equations. We consider the Lagrangean $\mathcal{L}$ locally invariant under the gauge group of transformations [1]

$$(1) \quad \mathcal{L} = \frac{1}{2} (\nabla_\mu \Phi)^x (\nabla_\mu \Phi) + \frac{1}{2} m^2 \Phi^x \Phi - \frac{1}{4} F_{\mu\nu} F_{\mu\nu},$$

$\nabla_\mu = \partial_\mu - \frac{ig}{\hbar} A_\mu$ is the covariant derivative, A_μ —the vector field, g —a constant of coupling, $\hbar$ —the Planck's reduced constant and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ the tensor of the electromagnetic field.

Using the cylindrical coordinates (r, θ, z) , where

$$(2) \quad \begin{aligned} \vec{U}_r &= \frac{\partial}{\partial r}, & \vec{U}_\theta &= \frac{1}{r} \frac{\partial}{\partial \theta}, & \vec{U}_z &= \frac{\partial}{\partial z}, \\ \nabla_r &= \frac{\partial}{\partial r}, & \nabla_\theta &= \frac{1}{r} \frac{\partial}{\partial \theta} - \frac{ig}{\hbar} A_\theta, & \nabla_z &= \frac{\partial}{\partial z}, \end{aligned}$$

$\vec{A} = A_\theta \vec{U}_\theta$ (for a magnetic field B_z),

$$F_{12} = -F_{21} = \frac{\partial A_\theta}{\partial r} + \frac{A_\theta}{r},$$

the others components of $F_{\mu\nu}$ vanish; the explicit form for $\mathcal{L}$ given by (1) reads

$$(3) \quad \begin{aligned} \mathcal{L} = \frac{1}{2} \left\{ \frac{\partial \Phi^x}{\partial r} \frac{\partial \Phi}{\partial r} + \frac{ig}{r\hbar} A_\theta \Phi^x \frac{\partial \Phi}{\partial \theta} + \frac{1}{r^2} \frac{\partial \Phi^x}{\partial \theta} \frac{\partial \Phi}{\partial \theta} + m^2 \Phi^x \Phi + \right. \\ \left. + \frac{g^2}{\hbar^2} A_\theta^2 \Phi^x \Phi - \frac{ig}{\hbar r} A_\theta \frac{\partial \Phi^x}{\partial \theta} \Phi - \left(\frac{\partial A_\theta}{\partial r} \right)^2 - 2 \frac{A_\theta}{r} \frac{\partial A_\theta}{\partial r} - \frac{A_\theta^2}{r^2} \right\}. \end{aligned}$$

We may use the Euler-Lagrange equations

$$(4) \quad \frac{\partial}{\partial r} \left(\frac{\partial L}{\partial A_{0,r}} \right) = \frac{\partial L}{\partial A_0},$$

$$(5) \quad \frac{\partial}{\partial r} \left(\frac{\partial L}{\partial \Phi_{,r}^x} \right) + \frac{\partial}{\partial \theta} \left(\frac{\partial L}{\partial \Phi_{,\theta}^x} \right) = \frac{\partial L}{\partial \Phi^x},$$

$$(6) \quad \frac{\partial}{\partial r} \left(\frac{\partial L}{\partial \Phi_{,r}} \right) + \frac{\partial}{\partial \theta} \left(\frac{\partial L}{\partial \Phi_{,\theta}} \right) = \frac{\partial L}{\partial \Phi},$$

where

$$(7) \quad L = \sqrt{g'} \mathcal{L}.$$

We consider the statical and z -independent distribution for the fields. In this case

$$(8) \quad \sqrt{g'} = r,$$

with L given by (7), (8) and (3); the Eqs. (4) and (5) become

$$(9) \quad \frac{\partial^2 A_0}{\partial r^2} + \frac{1}{r} \frac{\partial A_0}{\partial r} + \left(\frac{g^2}{\hbar^2} \Phi^x \Phi - \frac{1}{r^2} \right) A_0 = - \frac{ig}{2r\hbar} \left(\Phi^x \frac{\partial \Phi}{\partial \theta} - \frac{\partial \Phi^x}{\partial \theta} \Phi \right),$$

$$(10) \quad \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \Phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} - \frac{2ig}{r\hbar} A_0 \frac{\partial \Phi}{\partial \theta} - \left(m^2 + \frac{g^2}{\hbar^2} A_0^2 \right) \Phi = 0.$$

We choose for (10) a solution of form

$$(11) \quad \Phi = R(r) e^{in\theta},$$

which satisfy the angular part of Eq. (10); $R(r)$ is a finite function for $r \rightarrow \infty$.

Including (11) in (9) we get A : ($A = A_0$)

$$(12) \quad \frac{d}{dr} \left[\frac{1}{r} \frac{d}{dr} (rA) \right] + \left(A \frac{g^2}{\hbar^2} - \frac{ng}{\hbar r} \right) R^2 = 0.$$

We notice that

$$(13) \quad A = \frac{n\hbar}{gr}$$

is a solution for (12).

With Φ given by (11) and A given by (13), we try to find $R(r)$ from (10):

$$(14) \quad \frac{1}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) - m^2 R = 0.$$

One may see that (14) is a Bessel modified equation in the case $\nu=0$, so the origine smooth solution is

$$(15) \quad R(r) = I_0(mr).$$

But, for $r \rightarrow \infty$, this function grows up exponentially. In order to assure a finite limit for Φ (in case $r \rightarrow \infty$) we must write (14) as

$$(16) \quad \frac{1}{r} \frac{d}{dr} \left(r \frac{dR}{dr} \right) - (m^2 - fR^2)R = 0.$$

This equation can be obtained from a Lagrangean

$$(17) \quad \mathcal{L}' = \frac{1}{2} (\nabla_\mu \Phi)^* (\nabla_\mu \Phi) - \frac{1}{2} f \left(\frac{m^2}{f} - \Phi^* \Phi \right)^2 - \frac{1}{4} F_{\mu\nu} F_{\mu\nu},$$

with f is a constant of coupling.

But (17) is quite the Nielsen-Olesen Lagrangean for the spontaneous symmetry-breaking [1].

So, in order to satisfy the natural boundary conditions at infinite for the fields solutions A_0 and Φ , it was necessary a spontaneous symmetry-breaking.

2. The Quantized Hall Resistance. With the solution A given by (13) written as

$$(18) \quad A = \frac{n\hbar}{er},$$

where e is the electron's charge, we can calculate the magnetic flux through the plane (xy), namely

$$(19) \quad \Phi = \int_S \vec{B} \cdot d\vec{S} = \oint_{\Gamma} \vec{A} d\vec{l} = \int_0^{2\pi} \frac{n\hbar}{er} r d\theta = \frac{n\hbar}{e};$$

$$(20) \quad \Phi_0 = \frac{h}{e}$$

is the flux quantum.

It is known that the Hall potential difference U_H is

$$(21) \quad U_H = \frac{I\Phi}{Ne},$$

where I is the current, N — the number of carriers, Φ — the magnetic flux through the sample.

We name „Hall resistance“ the expression

$$(22) \quad R_H = \frac{U_H}{I} = \frac{\Phi}{Ne}.$$

Each flux line has m charge carriers attached on it, and

$$(23) \quad m = \frac{\Phi}{\Phi_0}$$

is the number of states per Landau level.

So the number N of the charge carriers (according to Pauli's principle) is

$$(24) \quad N = nm.$$

Considering (24), (22) can be written as

$$(25) \quad R_H = \frac{h}{me^2},$$

where R_H is the quantized Hall resistance and h/e^2 is the Hall resistance quantum.

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REFERENCES

1. Chaichian M., Nelipa N. F., *Introduction to Gauge Field Theories*. Springer Verlag, Berlin—Heidelberg—New York—Tokio, 1984.
2. Novacu V., *Teoria cuantică a cîmpului*. Ed. Tehn., Buc., 1984.
3. De Witt B. S., *Dynamical Theory of Groups and Fields*. Les Huches, 1969.

REZISTENȚA HALL ȘI RUPEREA SPONTANĂ DE SIMETRIE

(Rezumat)

Se poate obține cuantificarea rezistenței Hall folosind un Lagrangian gauge invariant (U_1) cu simetrie axială și rupere spontană de simetrie.

CHAOTIC EVOLUTION OF A RELATIVISTIC OSCILLATORY SYSTEM

BY

IULIAN ARCHIP, B. CONSTANTIN and P. SUFIȚCHI

1. Introduction. During the last years, the study of nonlinear phenomena and connected to this, the study of chaotic behaviour, has obviously constituted one of the most interesting and productive fields of investigation of the human knowledge. The researches on the oscillating nonlinear systems have taken into consideration systems where the nonlinearities are induced by the structures of the exterior fields [9], [11].

The purpose of the present study is to show that the variations of the mass with the velocity may constitute a new source of nonlinearity for an harmonic driving oscillator, which implies the possibility to set in the chaotic type evolution.

2. Stating the Pattern. We are going to consider, as follows, the unidimensional movement of a mass body $m=m_0(v)$ situated in a field of elastic powers, $U \sim x^2$. We also admit the existence of several dissipative forces $F_f \sim p$, p being the relativist impulse, and the presence of a periodical external excitation of pulsation ω .

The suggested form for F_f is justified by the fact that, taking into account the condition $v/c \rightarrow 0$, this term becomes the one already known for the force of viscous friction. Under these conditions, the movement equation is

$$(1) \quad \frac{dp}{dt} = -kx - \alpha p + F_0 \sin \omega t,$$

where $p = m_0 v (1 - v^2/c^2)^{-1/2}$, k and α are the coefficients of the elastic force, respectively — of the damping force, F_0 is the amplitude of the driving force of pulsation ω . Applying d/dt to the Eq. (1) there results

$$(2) \quad \frac{d^2 p}{dt^2} = -kv - \alpha \frac{dp}{dt} + \omega F_0 \cos \omega t.$$

Replacing $v = p(1 + p^2)^{1/2}$ and rescaling the impulse and the time to $P = p/m_0 c$, respectively to $\tau = t/2\pi T$; with $T_0 = 2\pi(m_0/k)^{1/2}$ the equation (2) may be re-written in the dimensionles form

$$(3) \quad \ddot{P} + P(1 + P^2)^{-1/2} + \gamma \dot{P} = \Omega f \cos \Omega \tau,$$

$$(\ddot{}) = \frac{d^2}{d\tau^2}, \quad (\dot{}) = \frac{d}{d\tau}, \quad \Omega = \frac{\omega}{\omega_0} > 0;$$

γ, f are undimensional parameters linked to the damping and driving forces by

$$\gamma = \frac{\alpha T_0}{2\pi m_0 c}, \quad f = \frac{F_0}{m_0 c \omega_0}.$$

Eq. (3) is equivalent to an nonautonomous system of ordinary differential equations 2D or (more convenient to a numerical treatment) by the introduction of a new variable $z = \Omega \tau$, with an autonomous system 3D

$$(4) \quad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x(1 + x^2)^{-1/2} - \gamma y + \Omega f \cos z, \\ \dot{z} &= \Omega, \end{aligned}$$

where $\dot{P} = \dot{x} = y$.

3. Particular Cases. i) When $\gamma = 0, f = 0, \Omega = 0$ which comes back to an evolution in the absence of the damping and forcing, system (4) is reduced to

$$(5) \quad \begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x(1 + x^2)^{-1/2}, \end{aligned}$$

which represents a hamiltonian system of energy

$$(6) \quad E \equiv H_1(x, y) = \frac{y^2}{2} + (1 + x^2)^{1/2}.$$

The phase plane (Fig. 1) indicates the existence of only one fixed point of an elliptical type; the progress in time is equivalent to the movement of a particle in a potential that has the form $(1 + x^2)^{1/2}$, the movement being periodical of pulsation with the data of [8]

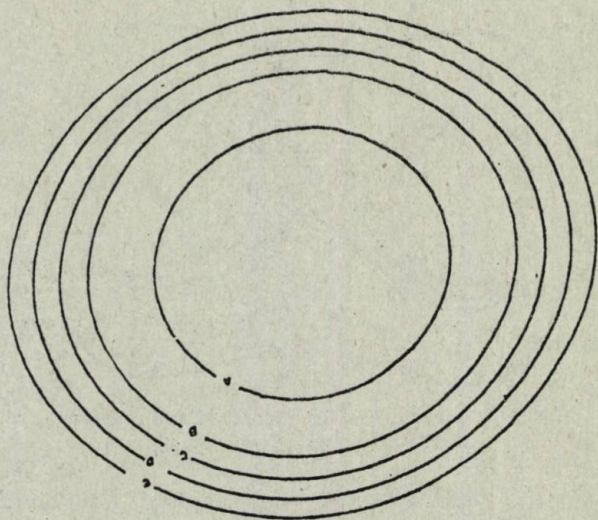
$$(7) \quad \omega = 2\pi \left[\oint \frac{dx}{\partial H_1 / \partial y} \right]^{-1}.$$

For $y = 0$, we have

$$(8) \quad H_1 = (1 + x_m^2)^{1/2}; \quad x_m = x_{\max},$$

$$(9) \quad \omega = \begin{cases} 1 - \frac{3}{16} x_m^2, & x_m \ll 1, \\ \pi(8x_m)^{1/2}, & x_m \gg 1. \end{cases}$$

Such an approach constitutes the starting point in the analysis of the phenomenon of breaking a wave of great amplitude in plasma as an effect of the variation $m = m(v)$, where the equations of the form (5) are derived for electrons in the acception of the model of cold fluid [12].

Fig. 1 — Level curves ; $H=H_1(x, y)=\text{const.}$

ii) We consider $f > 0$, $\gamma > 0$, $\Omega > 0$. If we admit that $P \ll 1$ (equivalent to the condition $v/c \ll 1$) the system (4) is that corresponding to the equation of an oscillator of Duffing type, periodically forced

$$(10) \quad \ddot{x} + \gamma \dot{x} + x - x^3 = f \cos \omega t.$$

Eq. (10) once more is present in the modelling of some interesting physical phenomena — the conduction of a particle in an anharmonic potential under the influence of a periodical external field (Hubermann and Crutchfield [6]), periodically forced beam buckled either mechanically or by magnetic field (Moon and Holmes [5]). Duffing equation is the subject of several detailed analytical and numerical investigations [2], [3], [5].

If γ and f are small enough, then the associated system may be regarded as a perturbation of a conservative system

$$(11) \quad H_2(x, y) = \frac{y^2}{2} + \frac{x^2}{2} - \frac{x^4}{4}.$$

The contour lines of the energy (Fig. 2) point the existence of three families of periodical orbits A, B, C, and of a double homoclinic orbit D. According to [4], the presence of the stable but nonintegrable elliptical fixed points O_1 , O_2 , and of the hyperbolic point O_3 , leads to the appearance of the chaotic evolution in the presence of the system's perturbations. R. van Dooren [3] confirms the inducement of the chaos characterized by the presence of a strange attractor (and complementary of a continuous band in the power spectrum), beginning with a threshold value $\Omega = 0.528$.

4. The Numerical Analysis. The pattern suggested by us, in form (4), belongs to the general category of the dissipative dynamic system 3D. The

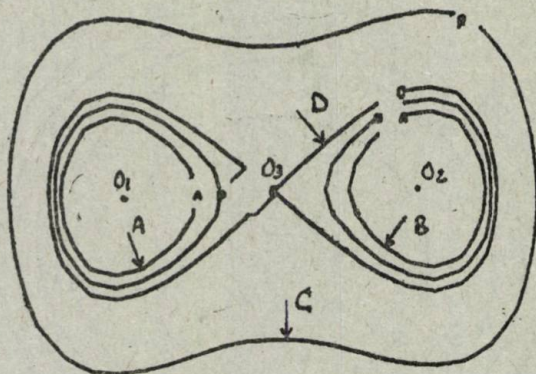


Fig. 2 — Level curves for undamped and unforced Duffing's system : $H_2 = H(x, y)$.

dissipative character is pointed out by the fact that the divergence of the velocities field defined by (4) is a negative value

$$(12) \quad \operatorname{div} \mathbf{v} = \sum \frac{dg_i}{dx_i}, \quad g_i = \dot{x}_i, \quad i = \overline{1, 3}.$$

When applying the theorem of divergence [1], we find out

$$(13) \quad \frac{dV}{dt} = \int_V \operatorname{div} \mathbf{v} d\Gamma,$$

$d\Gamma = dx dy dz$, the element of volume in phase space.

From (13) there results

$$\frac{dV}{dt} = -\gamma V,$$

that is $V = V_0 \exp(-\gamma t)$, which leads to the exponential of the phase flux.

The existence of the negative divergence constitutes an argument for the investigation of the system, taking into consideration the chaotic evolution [10]. Among the means of diagnosis the chaotic behaviour we have kept for the present analysis the study of the time series and the phase portrait [7].

The numerical experiments effected on the system of the form (4), experiments carried out on large sets of values of the control parameters γ, f , and Ω , point out a very riche and complex behaviour : periodical movement (in this case the attractor is of the limit cycle type), modulation period doubling, apparition of several phenomena of resonance and multiple resonants. At the same time there exists a large clase of time series which exhibits chaotic behaviour, aspect printed out by the portrait of phase.

Figs. 3...6 exemplify two such results, periodical and respectively chaotic series and the corresponding phase-space (projections in x, y , plane), indicating the values of parameters and the initial conditions.

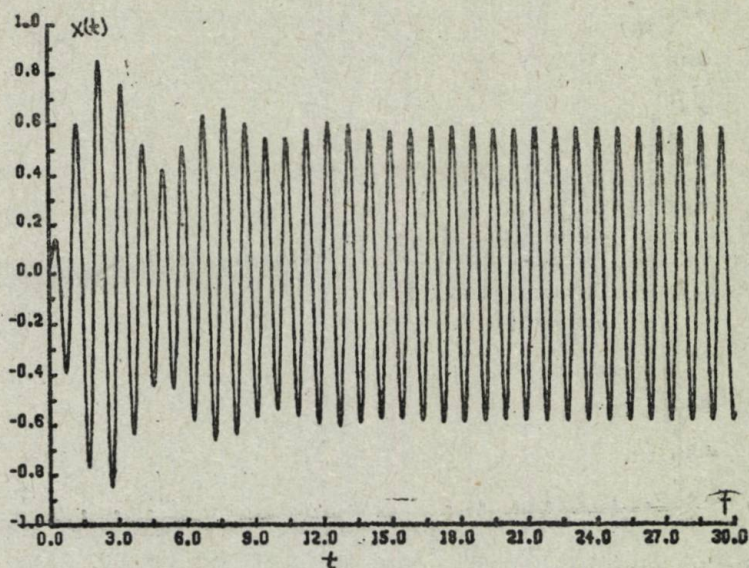


Fig. 3 — Time series of system (4); $\Omega=1.1$, $\gamma=0.15$, $f=1.1$, $x_0=0.03$, $y_0=0$, $z_0=0$.

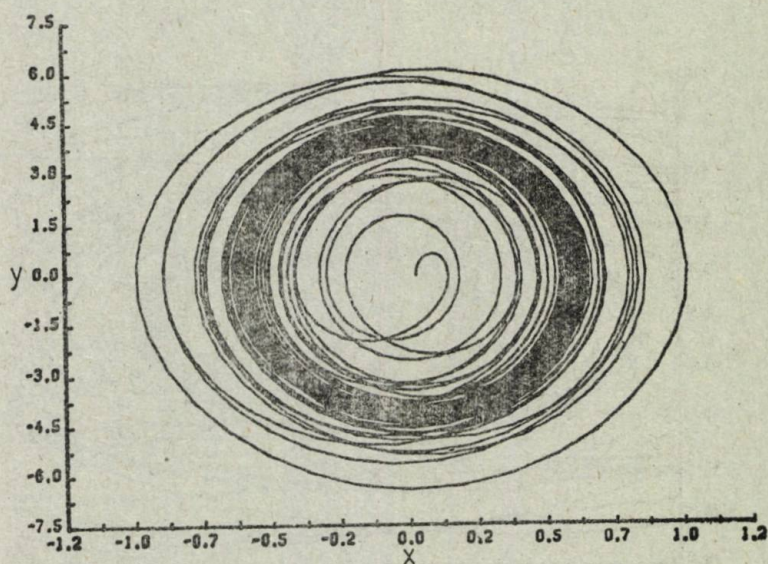


Fig. 4 — Periodical attractor of limit-cycle type corresponding to time series performed in Fig. 3.

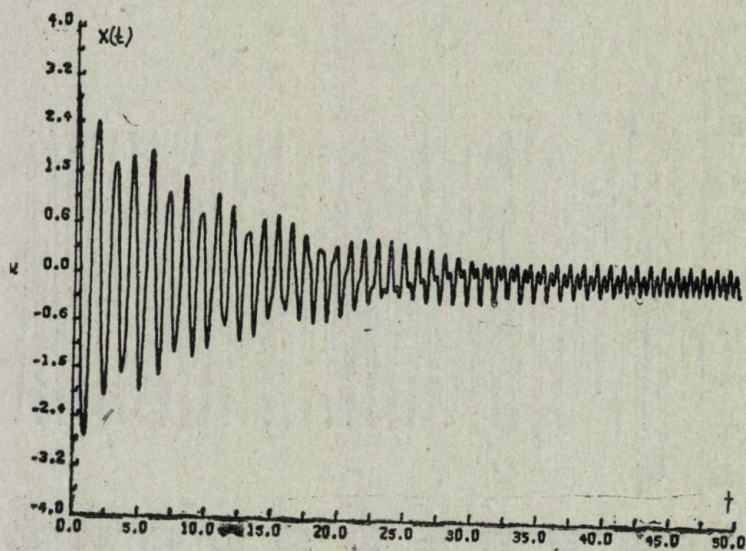


Fig. 5 — Chaotic time series of system (4); $\Omega=2$, $\gamma=0.15$, $f=2$, $x_0=3$, $y_0=0$, $z_0=0$.

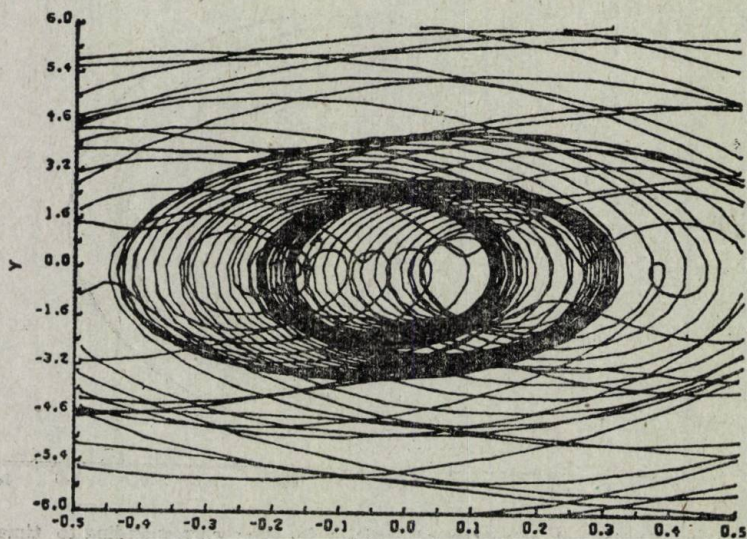


Fig. 6 — A region of phase space for system (4) corresponding to Fig. 5.

5. Conclusions. The analysed system implies a varied spectrum of behaviours which makes it fitted for the modelling of several real physical phenomena, thus improving the further study of the stability of the solutions, the accurate determination of the values of the parameters where the set in of the chaotic evolution takes place. We found also a new class of nonlinearities for physical dynamical systems.

Note. The numerical experiments were done in the Laboratory of Computational Physics of the Polytechnic Institute, Jassy.

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REFERENCES

1. Arnold V. I., *Metodele matematice ale mecanicii clasice*. E.S.E., Buc., 1980, p. 320.
2. van Dooren R., *On the Transition from Regular to Chaotic Behaviour in the Duffing Oscillator*. J. of Sound and Vib., **123**, 327 (1988).
3. van Dooren R., *Period Doubling Solution in the Duffing Oscillator. A Galerkin Approach*. J. of Comput. Phys., **32**, 1 (1989).
4. Hirooka H., Saito N., Ford J., *Chaos around Hyperbolic Fixed Points*. J. of the Phys. Soc. of Japan, **53**, 3, 895 (1984).
5. Holmes P., Whitley D., *On the Attracting Set for Duffing Equation. (I) Analytical Methods for Small Force and Damping*. In "Partial Differential Equations and Dynamical Systems", London, 1984, p. 211.
6. Huberman B. A., Crutchfield J. P., Phys. Rev. Lett., **43**, 1743 (1989).
7. Moon F. C., *Experiments in Chaotic Dynamics*. Courses and Lectures, **298**, CISM, Udine, p. 3.
8. Lichtenberg A. J., Lieberman M. A., *Regular and Stochastic Motion*. Appl. Math. Sci., Springer Verlag, 1983, p. 24.
9. Hao Bai-Lin, *Chaos*. W. S., Singapore, 1984.
10. Ott E., *Strange Attractors and Chaotic Motion of Dynamical Systems (I)*. Rev. of Mod. Phys., **53**, 4, 655 (1981).
11. Sagdeev R. Z., Usikov D. A., Zaslavskii G. M., *Nonlinear Physics; From Pendulum to Turbulence and Chaos*. H.A.P., 1988.
12. Shukla P. K., Rao N. N., Yu M. Y., Tsintsadze N.L., Phys. Rep., N. H., **138**, 50 (1986).

EVOLUȚIA HAOTICĂ A UNUI SISTEM OSCILANT RELATIVIST

(Rezumat)

Se studiază soluțiile unui sistem oscilant relativist, sursa neliniarităților constituind-o variația masei cu viteza. Utilizând metodele specifice fizicii neliniare se evidențiază dependența severă a caracteristicilor mișcării de condițiile inițiale și de parametrii de control. Pentru un set de valori sistemul evoluează haotic.

DETERMINATION OF DIPOLAR RELAXATION SPECIFIC PARAMETERS BY THERMALLY STIMULATED DEPOLARISATION CURRENT METHOD FOR POLYMERS PRESENTING LIQUID CRYSTAL PROPERTIES

BY

RODICA NEAGU, EUGEN NEAGU, I. I. NEGULESCU and W. H. DALY

1. Introduction. Four compounds of the polymer poly (γ -stearyl- α -L glutamate)s have been studied by thermally stimulated depolarisation current method. The presence of two types of units: units which participate in the formation of backbone of the main chain and units from which the side chains are constructed, is the most important property of these polymers. This duality of structure of the poly (γ -stearyl- α -L-glutamate) macromolecule results in the dual character of the ordering and properties of the system as a whole.

To obtain a thin polymer layer for electret preparation the four polymer samples have been solved in chloroform or in toluen. Thin polymer layer have been obtained from solution on aluminium discs. On the other face of the layer a second aluminium electrode was vacuum evaporated. Measurements have been made in a special cell in vacuum [1]. All obtained thermograms have a β -type maximum around -88°C . This maximum is determined by the relaxation of the dipoles with the smaller activation energy. Two other peaks of α -type were obtained.

The α_1 peak is specific to relaxation of main chain of mesogens unities and we believe that the temperature at which it appears is related with the temperature corresponding to T_g transition. This peak appears only if the sample was cooled down slowly. In this situation it is possible that the crystalline phase of the polymer be formed. If the sample is cooled down with big cooling rate, the α_1 peak is not obtained (Fig. 1) because now the conditions are not favourable for the crystalline phase apparition, but only for amorphous phase apparition. We believe that the α_2 peak is determined by the relaxation of mesogens unities and we consider that this peak appears at the same temperature as the transition of the polymer to liquid crystal state.

For all the studied polymers the activation energy and the relaxation time were determined using the initial rise method of Bucci—Fieshi and Guidi [2].

2. Experimental Results and Discussion. The experimental procedure was described in an other paper [3].

The BFG method of the initial rise consists briefly in that the time variation of electret polarisation is

$$P(t) = \epsilon_0(\epsilon_s - \epsilon_\infty) EF \exp \left(-s \int_{T_d}^T \alpha(T) dT \right),$$

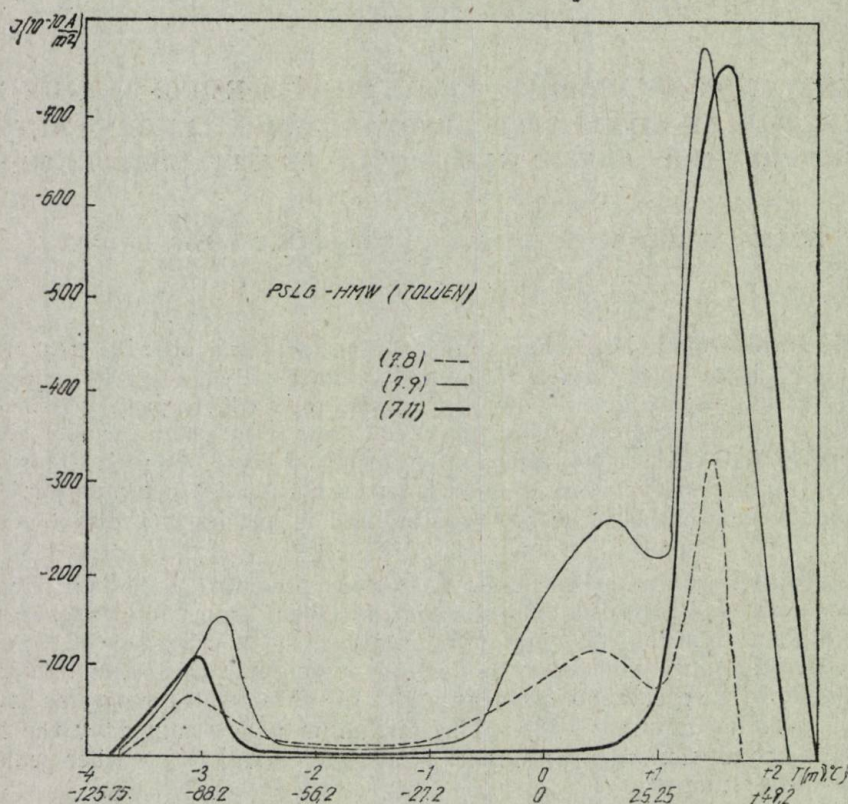


Fig. 1

where E is the applied field, F is the filling and storage factor, $s = (dT/dt)^{-1}$, $\alpha(T) = \alpha_0 \exp(-W/kT)$ (or $\tau = \tau_0 \exp(W/kT)$ is the relaxation time dependence on the temperature) and T_d is the initial temperature reached by the electret at the beginning of the CDST experiment. We note

$$P_0 = \epsilon_0(\epsilon_s - \epsilon_\infty) EF,$$

the maximum value of the electret polarization. The $P(t)$ variation determines a current density

$$j(t) = \frac{dP(t)}{dt} = -\alpha(T)P(t).$$

For the initial rise of the current we have

$$j(t) = \frac{P_0}{\tau_0} \exp \left(-\frac{W}{kT} \right)$$

and

$$(1) \quad \ln j(t) = -\frac{W}{kT} + \ln \frac{P_0}{\tau_0}.$$

From Arrhenius equation for the relaxation time and from the relation between peak maximum temperature and relaxation time

$$T_m = \frac{\tau(T_m)W}{ks}.$$

For τ_0 it results

$$(2) \quad \tau_0 = \frac{T_m^2 ks}{W} \exp\left(-\frac{W}{kT}\right).$$

By using the relations (1) and (2) we can calculate the activation energy and the relaxation time for all the three maxima mentioned before.

In Figs. 2 and 3, the dependence of logarithm of current density $\ln j(t)$ versus $1000/T$, for β type maxima, are represented. From both figures we see that straight lines have been obtained for all studied compounds. All the activations energies are sufficiently well grouped around the value of 0.196 eV. From Table 1 we can conclude that this relaxation process is common for all the studied compounds and is determined by the movement of the smallest polar groups, that is those which require the smallest activation energy for relaxation.

In Figs. 4 and 5 the dependence of logarithm of current density $\ln j(t)$ versus $1000/T$ for α_1 and α_2 type maxima are represented.

Table 1

The β peak

Sample	s^{-1} °C/min	T_m °C	U_{pol} V	T_{pol} °C	t_{pol} n	W eV	τ_0 s	Observations
V-61-M	2.3	-86.5	30	20	12.5	0.183	$3.88 \cdot 10^7$	first determination
IV-123-EX	4.25	-83.5	30	20	10.5	0.330	$1.037 \cdot 10^{11}$	first determination
IV-36-								
200 : 100	2	-90	30	20	29	0.205	$2.198 \cdot 10^8$	first determination
PSLG-HMW (CHCl ₃)	1.8	-87	30	20	15	0.233	$1.000 \cdot 10^9$	first determination
	3	-80	30	20	7	0.247	$7.402 \cdot 10^8$	first determination (7.1)
	3.2	-86	30	20	7	0.202	$8.866 \cdot 10^7$	quenched (7.3)
	3.2	-89	60	20	11.5	0.154	$3.400 \cdot 10^5$	quenched (7.4)
	2.5	-83	62	20	6	0.184	$1.960 \cdot 10^6$	slow cooling (7.5)
PSLG-HMW (toluen)	2.1	-88	62.6	20	240	0.198	$1.070 \cdot 10^8$	slow cooling (7.6)
	2.08	-92	80	20	20	0.263	$9.600 \cdot 10^7$	slow cooling (7.7)
	0.85	-95	83	20	20	0.164	$5.018 \cdot 10^7$	slow cooling (7.8)
	2.83	-81	83	20	19	0.156	$5.710 \cdot 10^6$	slow cooling (7.9)
	1.23	-88	83	100	20	0.160	$1.730 \cdot 10^7$	slow cooling (7.10)
	2.23	-88	83	20	20	0.309	$9.600 \cdot 10^{10}$	quenched (7.11)
Mean values		-88				0.196	10^7	

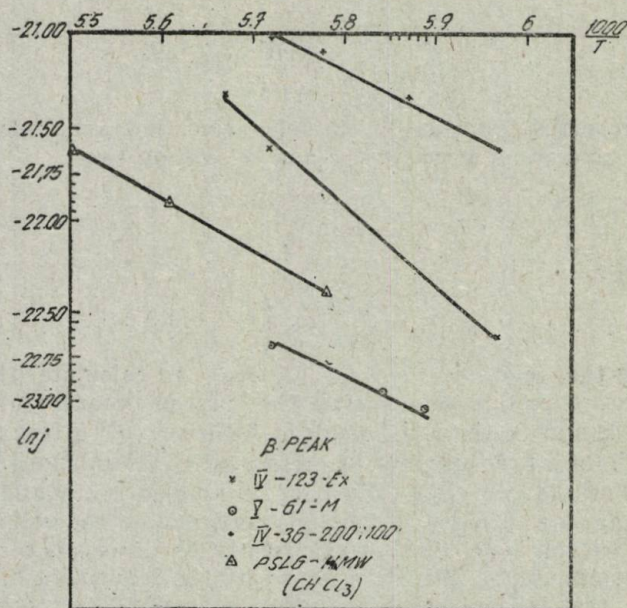


Fig. 2

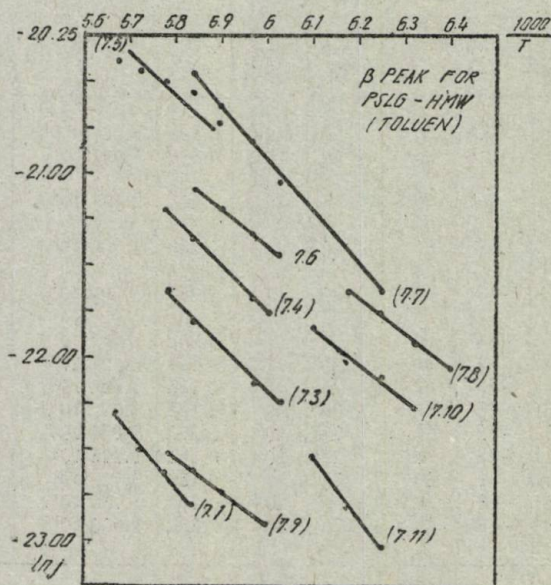


Fig. 3

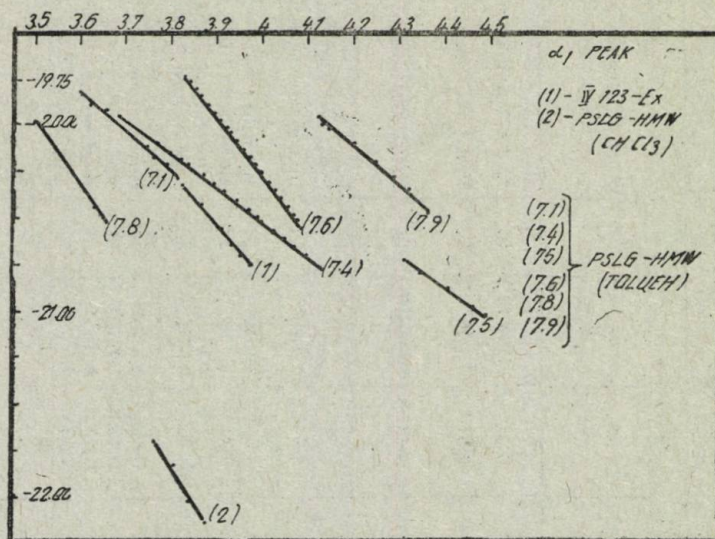


Fig. 4

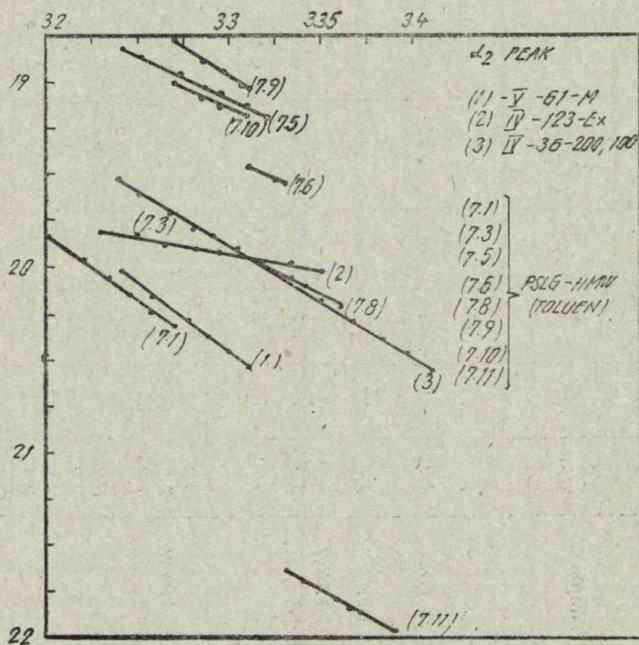


Fig. 5

Table 2
The α_1 and α_2 peaks

Sample	s^{-1} $^{\circ}\text{C}/\text{m}$	U^{pol} V	T^{pol} $^{\circ}\text{C}$	t^{pol} h	α_1			α_2			Observations
					T^{max} $^{\circ}\text{C}$	W eV	τ_0 s	T^{max} $^{\circ}\text{C}$	W eV	τ_0 s	
V-61-M	1	30	20	12.5	5	shoulder		40	0.51	$2.108 \cdot 10^{11}$	first determination
IV-123 EX	4.7	30	20	10.5	4	0.211	$3.45 \cdot 10^6$	48	0.31	$3.1 \cdot 10^7$	first determination
IV-36- 200 : 100	2	30	20	29	14	shoulder		42	0.54	$1.2 \cdot 10^{11}$	first determination
PSLG-HMW (CHCl_3)	3.3	30	20	15	21	0.36	$9 \cdot 10^8$	37	—	—	first determination
	3	30	20	7	19	0.175	$8.9 \cdot 10^5$	44	0.597	$1.2 \cdot 10^{12}$	first determination
	3.2	30	20	7	—	—	—	37	0.532	$1.4 \cdot 10^{11}$	quenched (7.3)
	3.2	30	20	20	—	—	—	40	0.214	—	quenched (7.4)
	2.5	62	20	6	19	0.183	$1.9 \cdot 10^6$	45	0.354	$2.14 \cdot 10^8$	slow cooling (7.5)
	1.93	62	20	240	7	0.221	$8.6 \cdot 10^6$	37	0.427	$2.07 \cdot 10^8$	slow cooling (7.6)
	2	80	20	20	17	shoulder		43	—	—	slow cooling (7.7)
	0.93	83	20	20	13	0.323	$6.6 \cdot 10^8$	37	0.585	$3.8 \cdot 10^{12}$	slow cooling (7.8)
	2.73	83	20	19	19	0.193	$2.3 \cdot 10^6$	37	0.436	$1.74 \cdot 10^8$	slow cooling (7.9)
	2.5	83	20	100	18	shoulder		43	0.43	5.10^8	slow cooling (7.10)
	2.4	83	20	20	—	—	—	45	0.43	$4.5 \cdot 10^9$	quenched (7.11)
Mean values					19	0.238	10^6	41	0.46	10^8	51

Table 1 and Table 2 show that the activation energy increases for the relaxation process taking place at greater temperatures. The liquid state begins approximately at the temperature of 41°C. For temperatures greater than $T_1=51^\circ\text{C}$ the orientation determined by the external electric field during polarisation is destroyed and starts the spontaneous orientation of polar groups, orientation that is specific to liquid crystal state. It is difficult to put into evidence this relaxation because the polymer layer becomes liquid and is destroyed under small mechanical pressure in the measuring cell. In paper [3] was presented a maximum obtained for the sample V-61-M at the temperature of 72°C. We consider that at this temperature the spontaneous orientation of the majority mesogenic unities takes place.

3. Conclusions. The obtained thermograms for the four studied samples have put into evidence the existence of three maxima:

i) The β peak which appears around -88°C and is determined by the relaxation of the polar groups which requires the smallest activation energy $W=0.196\text{ eV}$. The position of this peak is dependent on heating rate.

ii) The α_1 -type peak which appears in the samples cooled down slowly, namely in a wider temperature range around 19°C . Sometimes this peak appears as a shoulder of the α_2 -peak. The activation energy is around 0.24 eV .

iii) The α_2 -type peak which appears in all the samples no matter the way they were cooled. This peak appears around 41°C and the temperature range for this maximum is small. We consider that this maximum characterizes the melting behaviour of parafinic segments of side chains from poly (γ -stearyl- α -L-glutamate) as well as the onset of the LC state in these polymers.

Our results are in good agreement with the DSC thermograms [4].

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REFERENCES

1. Simionescu B., Neagu E., Leancă M., Polym. Bul., B **15** (1982).
2. van Turnhout J., *Thermally Stimulated Discharge of Electrets*. Elsevier, 1977.
3. Neagu E., Daly W. H., Negulescu I. I., Neagu R., Anal. Univ. Iași (1991).
4. Negulescu I. I., Daly W. H., Russo P. S., Poche D. S., Neagu R., Neagu E., Kraus M. L., Rev. Roum. de Chim. (to be published).

DETERMINAREA PRIN METODA CURENTULUI DE DEPOLARIZARE STIMULAT TERMIC A PARAMETRILOR SPECIFICI RELAXĂRII DIPOLARE LA POLIMERI CU PROPRIETĂȚI DE CRISTALE LICHIDE

(Rezumat)

Se studiază prin metoda curentului de depolarizare stimulat termic patru compuși ai polimerului poli (γ -stearyl- α -L-glutamat) care diferă între ei prin masa moleculară. Toate termogramele obținute prezintă un maxim de tipul β în jurul temperaturii de -88°C , determinat de mișcările dipolilor laterali. S-au obținut încă alte două maxime: maximul α_1 , specific relaxării lanțului principal și maximul α_2 , produs de relaxarea unităților mezogene. Considerăm că acest maxim apare la aceeași temperatură ca și starea de cristal lichid. Pentru toate maximele s-au determinat energia de activare și timpul de relaxare.

**MODIFICATIONS STRUCTURELLES INDUITES PAR LE
TRAITEMENT THERMIQUE DANS DES COUCHES MINCES
EN $\text{Bi}_2\text{Te}_{2,7}\text{Se}_{0,3}$**

PAR

EMIL LUCA, ALEXANDRINA JEFLEA, L. CRAUS et IRINA ANTONESCU

Les difficultés de l'obtention des couches minces en matériaux contenant des atomes de Se, dont les températures de dissociation sont voisines à la température d'évaporation, ont conduit au remplacement des méthodes thermiques par la méthode de la pulvérisation cathodique. La pulvérisation (sputtering) c'est le résultat du bombardement de la surface d'un matériau cible accompagné par l'émission d'électrons. Si l'émission est provoquée par le bombardement avec des ions positifs elle est nommée „cathodique“. Les atomes éjectés peuvent condenser sur un support et peuvent former une couche mince. Ce phénomène est connu depuis 1852 [1] et a été employé à la déposition des couches minces. Vu l'employ d'une décharge à pression élevée, la sensibilité à la contamination est grande et par suite ce procédé est considéré „sale“ ou impurificateur. Pourtant les technologies améliorées et les dispositifs perfectionnés de sputtering ont conduit à une révision de la position vis-à-vis de cette méthode, devenue à présent flexible et même plus pure que les autres. Les avantages de la pulvérisation cathodique sont :

a) la possibilité du transfert du matériau directement de la source semi-conductrice solide, non-chauffée, donc sans employer une nacelle qui pourrait participer à la réaction ;

b) le transfert de la substance semi-conductrice du cathode sur la couche croissante.

La méthode de la pulvérisation cathodique dépend de :

a) l'énergie des ions incidents (une dépendance spécifique) ;

b) le rapport des masses des ions incidents et des atomes de la cible ;

c) la chaleur de vaporisation du matériau de la cible.

Pendant la pulvérisation cathodique les particules éjectées (atomes ou conglomerats d'atomes) ont des énergies considérables et respectent approximativement la distribution maxwellienne, leurs vitesses étant plus grandes que celles des atomes évaporés. En quittant la surface du cathode, les atomes se trouvent dans des états excités et une émission de lumière caractéristique au phénomène de recombinaison apparaît.

Une méthode simple de sputtering est celle de la décharge luminescente de la diode. Les valeurs optimales de la pression pendant la pulvérisation ca-

thodique en décharge luminescente sont comprises entre 25...75 mtorr. En sachant que l'épaisseur d de l'espace obscur cathodique se trouve dans un rapport d'inverse proportionnalité avec la pression du gaz (la loi de Paschen), le produit pd est de 0,3 torr.cm au cas de l'argon [2].

Dans l'installation de laboratoire, de type HBA-1, la pulvérisation cathodique a été accomplie en atmosphère d'argon (l'énergie d'ionisation étant 15,76 eV/atome). Les matériaux de type $\text{Bi}_2\text{Te}_{3-x}\text{Se}_x$ ont une énergie de formation de 13,5...36,4 kcal/mole [3]. Dans [4] on trouve que la tension nécessaire à la pulvérisation cathodique du Bi_2Te_3 doit dépasser 1,5 kV à des pressions de $8 \cdot 10^{-2}$ torr. Ces conditions font possible l'ionisation de l'argon, la destruction des liaisons dans le matériau pulvérisé et l'accélération des conglomerats en vue de leur condensation sur le support.

Le schéma du dispositif de sputtering est représenté dans la Fig. 1 [5]. Le matériau à évaporer est placé sur le cathode. Toutes les pièces de raccord ont été écranées avec des conducteurs mis à la terre et placés à des distances moindres que l'espace obscur cathodique. Le matériau à pulvériser est en forme de disque coupé par électroérosion à fil, d'un lingot en $\text{Bi}_2\text{Te}_{2,7}\text{Se}_{0,3}$. Pendant la pulvérisation cathodique on contrôle la température du support à l'aide d'un thermocouple cuivre-constantan.

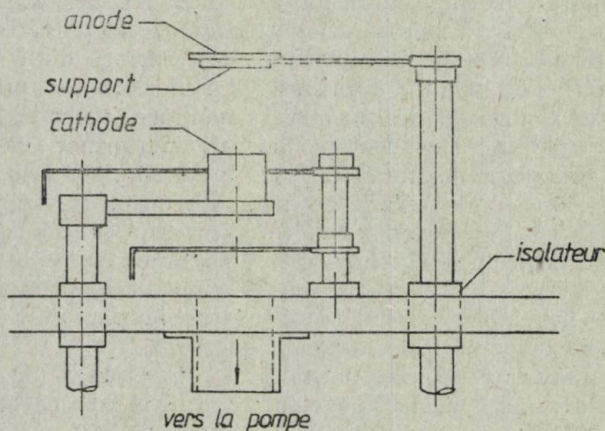


Fig. 1

De la littérature [4] il en résulte qu'à partir des températures du support supérieures à 100°C , Bi_2Te_3 ne respecte plus la stœchiométrie. Par conséquent, on a interrompu la décharge quand la température du support a atteint 100°C . Pendant le refroidissement, on vide l'installation jusqu'à $3 \cdot 10^{-5}$ torr, tant pour éviter l'impurification de la couche déposée, que pour la dégazer. Les conditions de pulvérisation sont : une tension d'environ 2,2 kV ; un courant de 65 mA ; la pression de l'argon $(3...5) \cdot 10^{-2}$ torr.

Dans ces conditions-ci la vitesse de croissance de la couche est de $5 \text{ \AA/s}$. Une déposition continue jusqu'à la température limite dure à peu près 15 minutes (l'épaisseur de la monocouche finale étant d'environ $0,45 \text{ \mu m}$). Des couches d'environ $5 \text{ \mu m}$ ont été obtenues par la continuation de la déposition après le refroidissement de la monocouche.

Les couches obtenues de telle manière sur des lames en verre nettoyées par décharge lumineuse sont très adhérentes et présentent des propriétés thermoélectriques satisfaisantes, les traitements thermiques qu'on leur applique ne conduisant qu'à la stabilisation de leurs propriétés.

Les couches y décrites ont été comparées avec le matériau massif de base ; de sorte trois catégories d'éprouvettes ont été analysées : „crues“, qui n'ont subies aucun traitement thermique, traitées thermiquement à 150°C et puis refroidies très lentement et le matériau massif de base.

L'analyse structurale fondamentale a été celle par diffraction des rayons X.

Les matériaux de type $\text{Bi}_2\text{Te}_{3-x}\text{Se}_x$ cristallisent dans le groupe spatial $\bar{R}3m$. Dans l'indexation hexagonale les constantes du réseau sont près de $4,3 \text{ \AA} = a$ et $30,5 \text{ \AA} = c$ [6].

La cellule hexagonale comprend 15 atomes. Les atomes de Bi — remplacés facilement par des atomes de Sb — peuvent occuper des sites de type

$$\left. \begin{array}{ccc} 0.000 & 0.000 & 0.396 \\ 0.000 & 0.000 & -0.396 \\ 0.333 & 0.667 & 0.063 \end{array} \right\} \text{Bi}_{0,5}\text{Sb}_{0,5} \times 3$$

$$\left. \begin{array}{ccc} 0.333 & 0.667 & 0.271 \\ 0.667 & 0.333 & 0.729 \\ 0.667 & 0.333 & -0.063 \end{array} \right\} \text{Sb}_1 \times 3$$

tandis que les atomes de Te ou de Se des sites de type

$$\text{Téllur (1)} \left\{ \begin{array}{ccc} 0.000 & 0.000 & 0.211 \\ 0.000 & 0.000 & -0.211 \\ 0.333 & 0.667 & 0.878 \\ 0.333 & 0.667 & 0.456 \\ 0.667 & 0.333 & 0.544 \\ 0.667 & 0.333 & 0.122 \end{array} \right\} \text{Te} \times 3$$

$$\text{Téllur (2)} \left\{ \begin{array}{ccc} 0.000 & 0.000 & 0.000 \\ 0.333 & 0.667 & 0.667 \\ 0.667 & 0.333 & 0.333 \end{array} \right\} \text{Te} \times 3$$

Ces atomes sont rangés dans 5 couches successives : téllur (1), bismuth, téllur (2), bismuth, téllur (1).

Entre les atomes de téllur il y a une liaison Van der Waals relativement faible, tandis qu'entre les atomes de bismuth et ceux de téllur (1) et, respectivement, téllur (2), il y a une liaison partiellement covalente, partiellement ionique.

Bi_2Te_3 et Bi_2Se_3 forment une solution solide dont les constantes du réseau sont : $a = 4,3 \text{ \AA}$ et $c = 30,3 \text{ \AA}$ [7], [8]. Jusqu'à la composition $\text{Bi}_2\text{Te}_2\text{Se}$, les atomes de Se occupent seulement les sites Te(2) et ensuite les sites Te(1). Les traitements thermiques de ces matériaux conduisent à des modifications

dans l'arrangement sur les sites destinées aux atomes de Se. La liaison Van der Waals entre les atomes de même sorte (Te—Te) est plus forte que celle d'entre les atomes différents (Te-Se).

Certaines études de diffraction des rayons X ont mis en évidence des modifications de l'intensité intégrale des lignes (322), (332) et (444) au cas des éprouvettes trempées par rapport aux éprouvettes refroidies lentement. Ces dernières ont une structure semblable à celle des éprouvettes avec des atomes de Se sur les sites du Te(2).

Au cas du matériau massif à la même formule on a pu mettre en évidence des modifications importantes de la constante du réseau déterminées par la température de traitement thermique.

Tant les couches minces déposées sur des lames en verre massif que le matériau de référence ont été étudiées à l'aide d'un diffractomètre DRON-20 à radiation Fe K α filtrée. Les éprouvettes n'ont subies aucune préparation supplémentaire.

Afin de déterminer les constantes de réseau ont été appliquées les corrections prévues dans l'ouvrage de Wilson [10]

$$(1) \quad \Delta\theta = -\frac{\sin 2\theta}{4\mu R} + \frac{t \cos \theta}{R \left(\exp \frac{2t\mu}{\sin \theta} - 1 \right)} - \frac{A^2 \sin 2\theta}{6R^2} - \frac{S \cos \theta}{R} - \frac{\delta \operatorname{ctg} 2\theta}{12},$$

où μ représente le coefficient d'absorption de l'éprouvette; R — le rayon du goniomètre; $t = 3,2 \sin \theta / \mu$; A — la largeur effective de l'éprouvette; S — la distance entre l'éprouvette et l'axe du goniomètre; δ — la divergence verticale du faisceau. Au cas du Bi₂Te_{2,7}Se_{0,3} on obtient

$$(2) \quad \Delta\theta = -(3,18 \cdot 10^{-4} \sin 2\theta + 3,18 \cdot 10^{-3} \cos \theta + 2,52 \cdot 10^{-6} \operatorname{ctg} 2\theta).$$

On peut remarquer que le terme le plus important est dû au déplacement de l'éprouvette par rapport à l'axe du goniomètre.

Au cas des angles larges, voisins de 180°, le troisième terme est prépondérant, à cause de la divergence du faisceau. Les erreurs peuvent être éliminées par une extrapolation en fonction de $\cos^2 \theta$, des constantes de réseau calculées [10].

Afin de calculer la largeur de la ligne de diffraction on a employé une éprouvette en silicium bien calibrée, dont un des maximums est placé au voisinage de la ligne principale de notre éprouvette (à environ 35°). La demi-largeur physique β_f de la ligne a été déterminée par la formule

$$(3) \quad \beta_f^2 = \beta^2 - \beta_{et}^2,$$

où β_f représente la demi-largeur de la ligne; β_{et} — la demi-largeur de la ligne étalon et β — la demi-largeur de la ligne proprement dite.

Afin de déterminer les dimensions des cristallites on a employé la formule

$$(4) \quad \Delta = \frac{0,94 \lambda_{Fe}}{\beta_f \cos \theta}.$$

Toutes les mesures ont été effectuées à la température ambiante. Les résultats obtenus au cas des trois sortes d'éprouvettes sont présentés dans les Tableaux 1 et 2.

Tableau 1

La variation des distances interplanaires

h	k	l	Matériau massif	Couche mince "crue"	Couche mince traitée thermiquement
1	0	1	3,7650	—	—
1	0	1	3,2106	3,2107	3,2992
1	0,1	0	2,3690	2,3492	2,4401
1	0,1	1	2,2317	—	2,2760
1	0,1	5	—	—	2,0557
1	0	0	2,1863	—	—
1	1,1	5	1,6054	—	—
1	0,2	0	1,4825	—	—
2	0,1	0	1,4362	—	—

Tableau 2

La variation de l'intensité de quelques maximums

h	k	l	Matériau massif	Couche mince „crue"	Couche mince traitée thermiquement
1	0	5	100	100	100
1	0,1	0	28	25	45
1	0,1	1	6	—	—
1	1	0	18	—	—

Les dimensions des cristallites ont été de 209 Å au cas de la couche mince sans traitement thermique, et, respectivement, 368 Å au cas de la couche mince traitée dans des cycles successifs échauffement - refroidissement lent.

Les constantes du réseau ont les valeurs $a=4,46$ Å et $c=30,85$ Å au cas des éprouvettes non-traitées, et, respectivement, $a=4,40$ Å et $c=29,9$ Å au cas de celles traitées thermiquement.

Conclusions. 1. Le traitement thermique détermine un réarrangement des atomes de Te et de Se sur les sites du réseau ; il est possible que tous les atomes de Se passent sur les sites du Te(2). Ce passage est accompagné par la diminution des constantes de réseau.

2. Les traitements thermiques à basses températures ne modifient pas la concentration du Se dans la solution solide, fait qui pourrait conduire à une variation importante des valeurs des constantes du réseau [6]. Par contre, un traitement thermique à haute température les modifie [11].

3. Dans les éprouvettes traitées, les cristallites augmentent et par conséquent le degré d'ordre se voit amélioré, fait confirmé par l'intensité de la ligne la plus importante.

4. Les diagrammes des couches traitées sont assez semblables à celles des matériaux massifs de sorte qu'on peut confirmer la correctitude de la méthode employée pour l'obtention de la couche mince.

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BIBLIOGRAPHIE

1. Growe W. R., Phil. Trans. Roy. Soc., London, 142, 87 (1852).
2. Holland L., *Vacuum Deposition of Thin Films*. Chapman & Hall, Ltd., London, 1961.
3. * * * *Физико-химические свойства полупроводниковых веществ. Справочник.* Изд. Наука, Москва, 1979.
4. Francome M. N., Phil. Mag., 10, 989 (1964).
5. Jeflea Al., Thèse de doctorat, Université „Al. I. Cuza“, Jassy, 1990.
6. Гольцман В. М., Кудинов В. А., Смирнов Ж. А., *Полупроводниковые термоэлектрические материалы на основе Bi_2Te_3* . Изд. Наука, Москва, 1972.
7. Bekebrede W. R., Guentert O. J., J. Phys. Chem. Sol., 23, 1 023 (1962).
8. Miller G. R., Ghe-Yu-Li, Spencer C. W., J. Appl. Phys., 34, 1398 (1963).
9. Misra S., Bever M. B., J. Phys. Chem. Sol., 25, 1 233 (1964).
10. Wilson A. J. C., Rev. Scient. Instrum., 20, 831 (1950).
11. Jeflea Al., Melinte S., Voiosek Șt., Bul. Inst. Polit. Iași, XXXIV (XXXVIII), 1-4, s. I, 89-92 (1988).

MODIFICĂRI STRUCTURALE INDUSE DE TRATAMENTUL TERMIC ÎN PĂTURILE SUBȚIRI DE $\text{Bi}_2\text{Te}_{2,7}\text{Se}_{0,3}$

(Rezumat)

Se prezintă proprietățile filmelor subțiri de $\text{Bi}_2\text{Te}_{2,7}\text{Se}_{0,3}$ obținute prin pulverizare. Se analizează condițiile de obținere a acestor filme și instalațiile de laborator corespunzătoare. Rezultatele obținute prin difracție cu raze X se prezintă pentru filme subțiri „crude“, precum și pentru cele supuse unui tratament termic la 180°C. Modificările structurale rezultate sînt analizate în relație cu măsurătorile electrice efectuate asupra acestor filme.

PHOTOELECTRODES OF IMPURIFIED $\alpha\text{-Fe}_2\text{O}_3$ METHOD OF PREPARATION AND ELECTRICAL PROPERTIES

BY

SERGIU ISTRATE, MARIANA ISTRATE, EUGEN OSADEȚ and C. PASNICU

1. Introduction. Ferric oxide $\alpha\text{-Fe}_2\text{O}_3$ is a possible photoelectrod. At least two characteristics recommend it for this purpose: energy band gap is 2.2 eV, value placed in the interval 1.5...2.5 eV required for the semiconductors utilized as photoanodes [1]; the second characteristic is the good electrochemical stability. Since, in the case of photoelectrolysis, because the photocurrent is notable, the electrical resistance of the semiconductor and electrolyte respectively must not be too large, it was necessary the impurification, at a certain level, of $\alpha\text{-Fe}_2\text{O}_3$ in order to decrease its resistivity.

In this work we present a preparation method for the compound $\alpha\text{-Fe}_2\text{O}_3$, polycrystalline as well as an installation for the determination of the electrical conductivity and Seebeck coefficient respectively, as function of the temperature. The plots $\ln\sigma=f(10^3/T)$ and $Q=f(T)$ are discussed.

2. Experimental Method. 2.1. Method of Preparation. The ferric oxide Fe_2O_3 is formed when the iron is whole oxidated in air, oxygen or nitrogen at high temperature without reaching the temperature of dissociation [2]. Works on methods of $\alpha\text{-Fe}_2\text{O}_3$ preparation was performed by Chevallier [3], Salmon [4], Suchet [5]. Salmon [4] concluded that until maximum sintered temperature of 1050°C only $\alpha\text{-Fe}_2\text{O}_3$ is formed and up to 1200°C start to appear magnetite as distinct phase. In the range of sintered temperatures 1050...1200°C the magnetite is in solid solution with the hematite. The fact that in hematite can exist also magnetite is not a major impediment at least of two reasons:

- i) according to Vol [6] $\rho \approx 10^3 \Omega\text{m}$. The appearance of Fe_3O_4 in $\alpha\text{-Fe}_2\text{O}_3$ ensure the presence of ferrous ion Fe^{2+} who increases the electrical conductivity;
- ii) if the cooling is carried out slow enough, the Fe_3O_4 convert into $\alpha\text{-Fe}_2\text{O}_3$.

Suchet [5] analyses the dissociation when the hematite is impurified with compounds containing tetravalent cations for increasing the electrical conductivity. During the cooling, the gases of sintering atmosphere can penetrate in the material increasing the compound porosity. Consequently, the homogeneous structure obtained at high sintering temperatures will modify during the cooling in the oxidating atmosphere, but the rate of defects diffusion being small, this modification alters only the grain boundaries, especially the grains of the surface probe. The raw materials were the metallic oxides powders $\alpha\text{-Fe}_2\text{O}_3$, ZrO_2 , TiO_2 and Nb_2O_5 respectively.

For the purpose of obtaining a good chemical homogeneity in the sintered materials, it must be improved as far as possible the homogenization of mixture in the initial stage of preparation. For realization our probes we utilize as homogenization procedure the milling in a wet ball mill because $\alpha\text{-Fe}_2\text{O}_3$ was impurified with another oxides at small levels (1, 0.1, 0.01 at % respectively). The major problem is to avoid demixing following as a result of selective sedimentation; then the mixture water-powder was at once dried in a drying closet.

The powder realized as a result of the stages homogenization-milling, drying and crumbling, was dry pressed in form of rectangular prism, in which we introduced platinum wires in pre-established places for obtaining good electrical contacts; we utilize a hydraulic-press equipped with a manometer in order to establish the most favourable pressure and then to control it. We carry out different thermic treatments who are presented in Table 1, and we modify the sintering temperature, the bearing duration and the work atmosphere (air or nitrogen). We conclude that in case of impurified $\alpha\text{-Fe}_2\text{O}_3$, the treatments at 1 200°C in air are convenient if the cooling is slow in the furnace.

Table 1

Thermal treatments

Treatment	Sintering temperature °C	The bearing duration hours	The cooling
T ₁	1 200	24	slow in air
T ₂	1 250	6	
T ₃	1 300	6	
T ₄	1 200	16	
T ₅	1 200	8	
T ₆	1 200	9	

The structure of our probes was determined with the aid of DRON-2.0 diffractometer, utilising $\text{FeK}_{\alpha\beta}$ radiation. The obtained structure, results, according to „Inorganic Index to the Powder Diffraction File“ 1965 and card indes ASTM respectively, as being $R\bar{3}c$. It was not observed another phases in the materials sintered at 1 200°C.

2.2. The Installation for the Determination of the Electrical Properties.

We realized probes of the shape presented in Fig. 1, which have four arms: 1 and 4 are the current leads, and at the at ends 2 and 3 we determine the voltage drop for the calculation of Seebeck coefficient Q as well as the electrical conductivity σ . Both voltage drop and current intensity was determined by means of an accurate high-impedance voltmeter; the current intensity was measured by the voltage in a standard resistance R_{st} (Figs. 1 and 2).

The ratio between the voltage drop across a sample and the current through the sample gives the resistance between lateral arms, the conducti-

vity being calculated from this value and the known dimensions of the sample. The probes of the type shown in Fig. 1 are widely used for semiconductor work. In such samples the effective region of contact between current carrying and noncurrent carrying regions is in the sample material itself. This fact is particularly important for semiconductors, since injection or extraction of carriers is thereby minimized. Of course, at the main current contacts injection still remains a possibility. However, if the distance between the current electrodes and the lateral arms is several diffusion lengths, or more from the contacts, there will be little or no difficulty with injection.

The rectangular prism in which we introduced the platinum wires before pressing and sintering respectively, in the corresponding places to the arms 1, 2, 3, 4, was processed after sintering by means of a diamond disc and transformed into the shape presented in Fig. 1.

The block diagram of the installation is reproduced in Fig. 2. In a little furnace we introduce the sample on a probe-holder. The temperature in the furnace is regulated with the help of a thermoregulator. As sensitive element for thermoregulator we utilise a chromel-alumel thermocouple. Also chromel-alumel thermocouples for the temperature determination in sample was used. The electrical conductors for the current determination in, and voltage on sample, were on platinum wires, as the measure thermocouples were connected to an switch, connected to a multimeter MERATRONIK, type V 543.

We realized the thermoregulator utilising an electrical diagram with a thyristor as regulator element. By means of this thermoregulator we adjusted the temperature in the limit of one Kelvin. Seebeck coefficient Q was obtained measuring thermoelectric voltage by the arms 2 and 3 of the sample (Fig. 1), with the probe placed in the thermal gradient on the furnace axis. The thermocouples were attached by means of a fireproof putty (kaolin and sodium silicate) at 2 and 3 arms in sample-holder. The thermal gradient was comprised between 2...15 K and as corresponding temperature to Q we utilise the arithmetic mean of two temperatures registered by thermocouples. The measurements were carried out about 1 350 K.

Because the electrical resistance of the semiconductors varies strongly with the temperature, it was necessary to ensure a constant temperature in the whole of the sample. In order to do it we placed the sample-holder in a metallic enclosure realized by means of a Kanthal strap spiral in the middle of the furnace, where the thermal gradient is minimum. After obtaining the desired value of temperature we expected about ten minutes, so that the temperature become the same in the whole enclosure.

3. Results and Discussion. In Figs. 3, 4, and 5 we present the electrical conductivity ($\ln\sigma$) versus inverted temperature ($10^3/T$) for all our probes of $\alpha\text{-Fe}_2\text{O}_3$ impurified with Zr, Ti and Nb, respectively.

It can be seen from the figures that the values of $\ln\sigma$ decreases almost linearly with increasing reciprocal temperature up to a certain temperature,

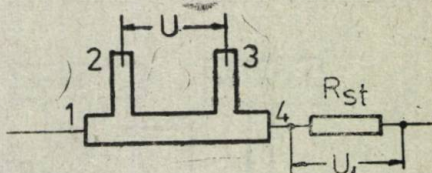


Fig. 1 — The shape of the sample.

where a change in slope occurs in the plots. These kinks correspond to exhaustion or saturation temperature of the impurity levels T_s , when all electrons of the donor levels are excited in the conduction band, and the temperature of the transition to intrinsic conductivity T_i is in the high temperature range. These

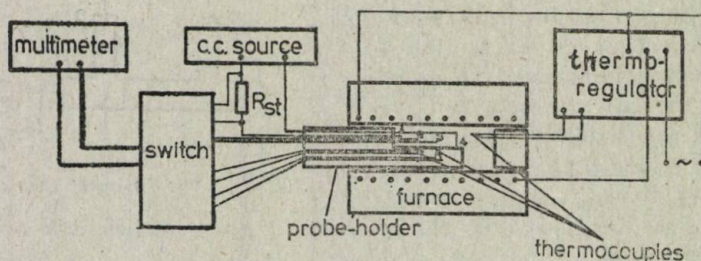


Fig. 2 — The installation for the determination of the electrical conductivity and Seebeck coefficient respectively, versus temperature.

temperatures are not observed in all our samples; then T_s appears only in the probes containing the impurity of 0.1 and 1% respectively, and T_i only in the doped samples with Zr and Ti respectively, with 0.1%. All the doped samples with 0.01% exhibit only one slope. The activation energies of the impurified $\alpha\text{-Fe}_2\text{O}_3$ have been calculated from the slopes of plots and are presented in Table 2.

Table 2

The transition temperatures T_s and T_i and the corresponding activation energies

Sample	T_s K	T_i K	E_1 eV	E_2 eV	E_3 eV
Zr 0.01					
Zr 0.1	500	1 000	0.99	0.45	2.28
Zr 1	360		0.45	0.31	0.7
Ti 0.01					
Ti 0.1	630	1 190	0.89	0.50	2.00
Ti 1	410		0.38	0.22	2.20
Nb 0.01					
Nb 0.1	430		0.70	0.42	1.80
Nb 1	400		0.46	0.26	

The value of the Seebeck coefficient Q at different temperatures has been calculated from the thermoelectric voltage by the arms 2 and 3 of the probe; in the Figs. 6, 7 and 8 we plotted the Seebeck coefficient versus temperature. The Seebeck coefficient decreases in the low temperature range, is constant in the medium temperature range and for the weakly doped samples (0.1 and 0.01 respectively) the Seebeck coefficient increases in the high temperature range.

The Seebeck coefficient is determined [7], [8] by the reciprocal carrier concentration, the change in the position of the Fermi level, the carrier mobility, the drag of the electrons by the phonons, its sign being that of the majority

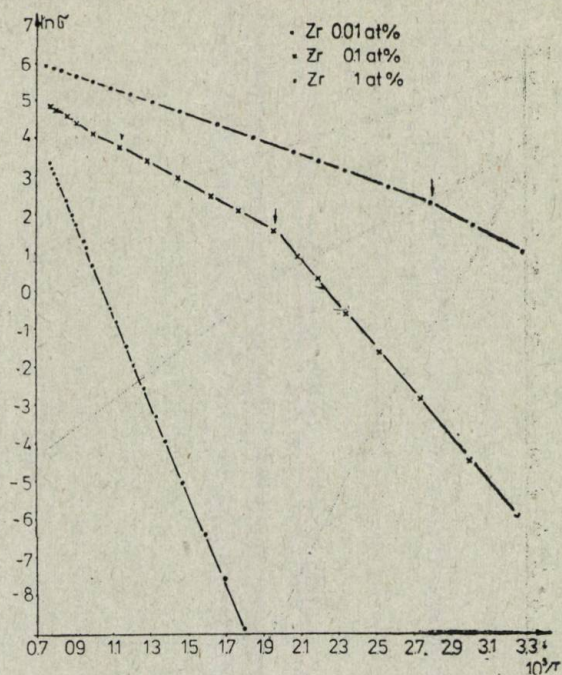


Fig. 3 — Electrical conductivity of $\alpha\text{-Fe}_2\text{O}_3$ impurified with Zr as a function of reciprocal temperature.

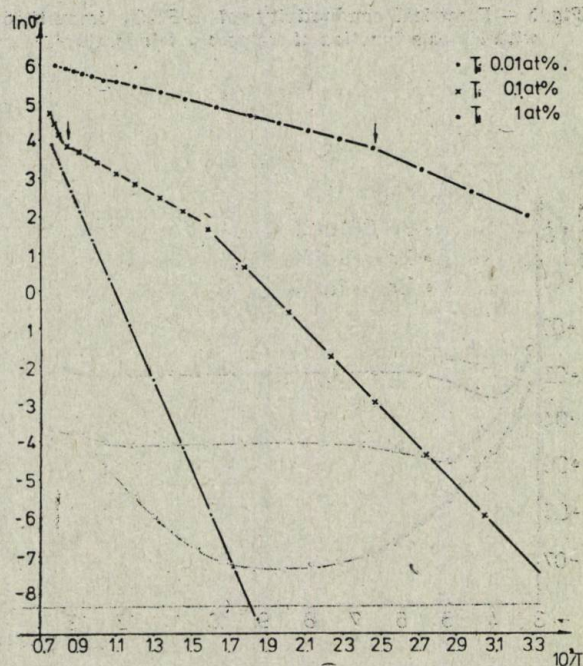


Fig. 4 — Electrical conductivity of $\alpha\text{-Fe}_2\text{O}_3$ impurified with Ti as a function of reciprocal temperature.

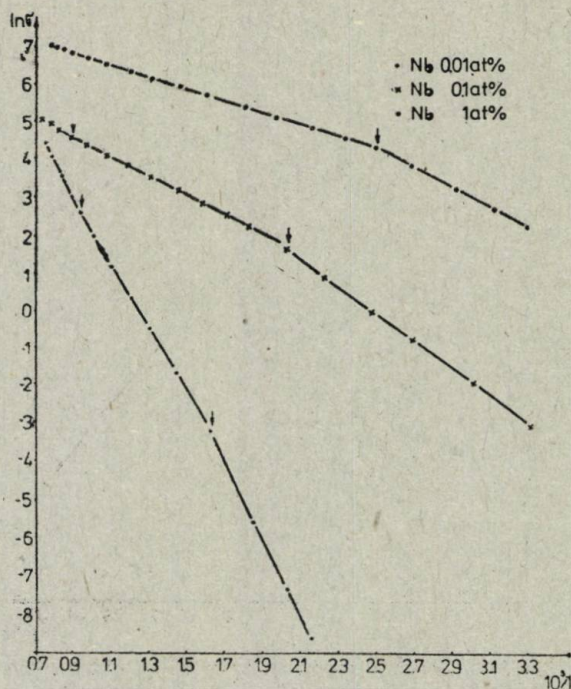


Fig. 5 — Electrical conductivity of $\alpha\text{-Fe}_2\text{O}_3$ impurified with Nb as a function of reciprocal temperature.

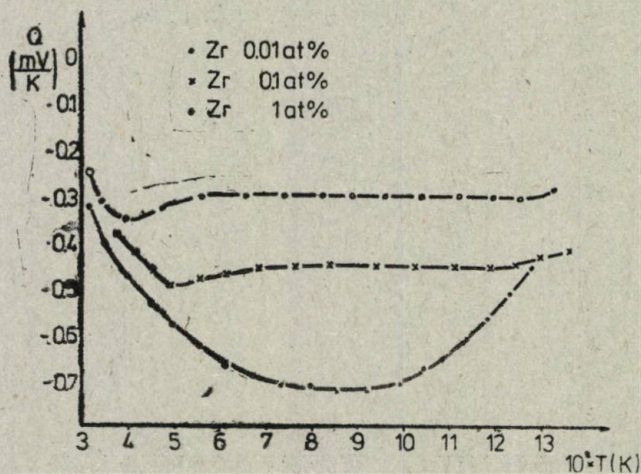


Fig. 6 — Seebeck coefficient of $\alpha\text{-Fe}_2\text{O}_3$ impurified with Zr versus temperature.

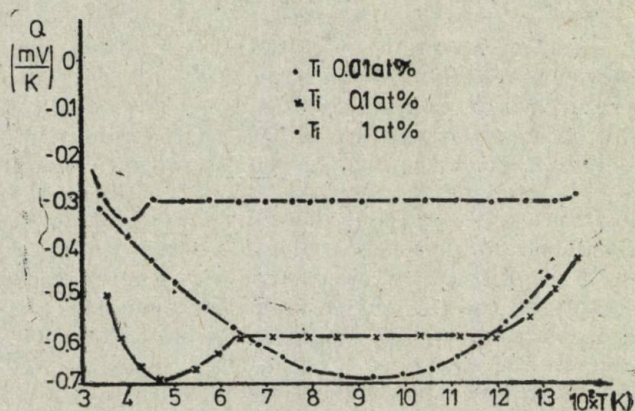


Fig. 7 — Seebeck coefficient of $\alpha\text{-Fe}_2\text{O}_3$ impurified with Ti versus temperature.

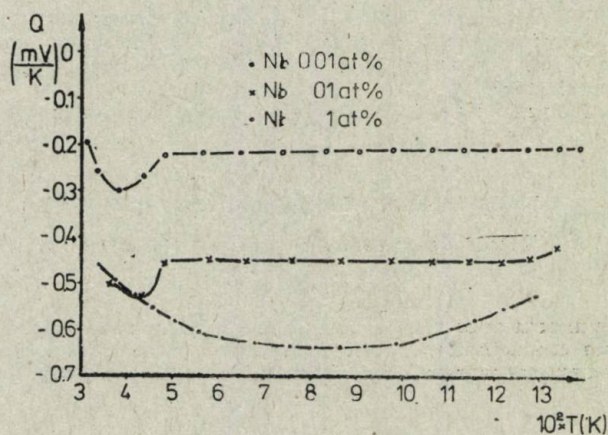


Fig. 8 — Seebeck coefficient of $\alpha\text{-Fe}_2\text{O}_3$ impurified with Nb versus temperature.

carriers. In the low temperature range the carrier concentration is small and then Q must be large ($\sim 1/n$); but because the great electrical resistance of the grain boundaries, the transport of the electrical charge is diminished. As the temperature increases the effect of the grain boundary decreases [9], [10] and then Q increases in spite of the fact that n rises as a result of the thermic activation from impurity atoms. If temperature rises again the concentration n increases as consequence of ionization impurities and then Q decreases. In the medium temperature range, at temperatures upper to exhaustion temperature T_s , but below to intrinsic temperature T_i , the carrier concentration n is constant and when the effect of the change in the position of the Fermi level is compensated by the growing of mobility with temperature, the Seebeck coefficient remains constant. At the high relative temperature, in case the samples are weakly doped, an intrinsic conduction appears, and then both types of carriers, n and p respectively are generated, and thus the electron concentration increases involving the diminution of Q . Moreover, the carriers of type p generation accompanies an opposite sign for Seebeck effect which annuls one part of Q_n — due to the electrons — because $Q_p > 0$ and $Q_n < 0$; then the Q variation is relative abrupt. This last conclusion is true only when the intrinsic bipolar conduction appears.

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REFERENCES

1. Mavroides J. G., Mat. Res. Bull., **13**, 1 379 (1979).
2. Michel A., *Sesquioxide de fer Fe_2O_3* . Dans le *Nouveau traité de chimie minérale* (Ed. P. Pascal), **XVII**, 1, Masson et C^{ie}, 1963, p. 636.
3. Chevallier R., Journ. phys. rad., **12**, 172 (1951).
4. Salmon O. N., J. Phys. Chem., **65**, 550 (1961).
5. Suchet J., Journ. phys. rad., Supl. no. 3, **18**, 10 A (1957).
6. Вол А. Е., *Строение и свойства двойных металлических систем*. Т. II, ГИФМЛ, Москва, 1962, стр. 667.
7. Epifanov G. I., *Solid State Physics*. Mir Publishers, Moscow, 1979.
8. Anselm A., *Introduction to Semiconductor Theory*. Mir Publishers, Moscow, 1981.
9. Gardner R. F. G., Sweett F., Tarner D. W., J. Phys. Chem. Solids, **24**, 1 175 and 1 183 (1963).
10. Maiti H. S., Rajendran S., Sitakara Rao, Phys. stat. sol., (a), **84**, 655 (1984)

FOTOELECTROZI DIN α - Fe_2O_3 IMPURIFICAT

Metoda de preparare și proprietăți electrice

(Rezumat)

Se prezintă o metodă de obținere a compusului α - Fe_2O_3 policristalin precum și o instalație pentru determinarea conductivității electrice, respectiv a coeficientului Seebeck funcție de temperatură. Sunt discutate reprezentările grafice $\ln \sigma = f(10^3/T)$ și $Q = f(T)$.

SUR DES NOUVEAUX ASPECTS CONCERNANT LA STABILITÉ DANS LE TEMPS DES PIÉZOCÉRAMIQUES À ACTIVITÉ PIÉZOÉLECTRIQUE INTENSE

PAR

IRINA ANTONESCU, M. ANTONESCU et ALEXANDRINA JEFLEA

1. Introduction. Les premiers résultats d'une étude systématique au sujet de la stabilité dans le temps des grandeurs caractéristiques d'une nouvelle céramique piézoélectrique, à activité intense [1], ont été présentés dans [2]. Ce sont eux qui ont imposé la suite de l'analyse d'autres traitements thermiques, ultérieure à celui de polarisation électrique, tant en vue de l'optimisation du vieillissement du matériau piézo-céramique étudié, que de la détermination des critères de prédiction de son évolution à long terme.

2. Expérimentation et discussions. Les éprouvettes, les mesures diélectriques et piézoélectriques, tout comme leur algorithme d'étude sont décrites dans les travaux [1] et [2]. Les conclusions des expériences présentées dans [2] ont exigé l'introduction de quelques nouveaux éléments expérimentaux, brièvement décrits ci-dessous.

En dehors des traitements de revenu, on a essayé aussi des traitements de trempe à partir des mêmes températures que celles de revenu. Les éprouvettes trempées, au lieu d'être refroidies lentement en thermostat, en ont été brusquement extraites, après avoir été maintenues, au préalable, pendant un quart d'heure, à la température de traitement. Dans le Tableau 1 comprenant un bref extrait des données expérimentales, les traitements de revenu ont été marqués par la lettre *L*, tandis que ceux de trempe par la lettre *R*, notées après la valeur, exprimée en degrés Celsius, de la température de traitement.

De plus, à 80°C, on a essayé deux types de traitements, A et B. Ils sont différents en ce qui concerne le moment de leur accomplissement : un jour (80 A) et, respectivement, dix jours (80 B) après le traitement de polarisation électrique. On a eu l'intention de préciser ainsi le sens de l'affirmation que „n'importe quel choc mécanique, électrique ou thermique conduit à la reprise du cycle de vieillissement“ [3].

Les coefficients de vieillissement pour décades, notés KT_y , où y exprime la valeur étudiée dans le temps, ont été calculés comme dans [2], pour dix-huit grandeurs d'intérêt, mesurées 1 jour, 10 jours et respectivement, 100 jours après le traitement subi, au cas des neuf traitements thermiques différents. La quantité de données expérimentales a réclamé leur traitement par ordinateur tant en ce qui concerne le calcul des grandeurs et des coefficients de

Tableau 1

Le traitement	$K T k_p$		$\Delta K T k_p$	Le traitement	$K T \epsilon_{33}^T$		$\Delta K T \epsilon_{33}^T$
	$J 1 \dots 10$	$J 10 \dots 100$			$J 1 \dots 10$	$J 10 \dots 100$	
23 L	-0,0076	-0,0018	-0,0058	23 L	-0,0323	-0,0061	-0,0262
60 L	0,0068	0,0021	0,0047	60 L	-0,0008	0,0143	-0,0151
60 R	0,0028	0,0023	0,0005	60 R	0,0002	0,0171	-0,0169
80 AL	-0,0008	0,0023	-0,0031	80 AL	-0,0324	-0,0054	-0,0270
80 AL	-0,0046	0,0002	-0,0048	80 AR	-0,0298	-0,0034	-0,0264
80 BL	0,0047	0,0051	-0,0004	80 BL	0,0078	0,0146	-0,0068
80 BR	0,0005	0,0028	-0,0023	80 BR	0,0065	0,0165	-0,0010
100 L	0,0021	0,0022	-0,0001	100 L	-0,0288	-0,0029	-0,0259
100 R	0,0027	-0,0064	0,0091	100 R	-0,0110	0,0023	-0,0133

	$K T d_{31}$		$\Delta K T d_{31}$		$K T g_{31}$		$\Delta K T g_{31}$
	$J 1 \dots 10$	$J 10 \dots 100$			$J 1 \dots 10$	$J 10 \dots 100$	
23 L	-0,0277	-0,0070	-0,0207	23 L	0,0048	-0,0009	0,0057
60 L	0,0060	0,0088	-0,0028	60 L	0,0068	-0,0053	0,0121
60 R	0,0028	0,0110	-0,0082	60 R	0,0026	-0,0059	0,0085
80 AL	-0,0190	-0,0019	-0,0171	80 AL	0,0139	0,0036	0,0103
80 AR	-0,0223	-0,0031	-0,0192	80 AR	0,0078	0,0003	0,0075
80 BL	0,0084	0,0125	-0,0041	80 BL	0,0000	-0,0023	0,0023
80 BR	0,0027	0,0108	-0,0081	80 BR	-0,0037	-0,0055	0,0018
100 L	-0,0143	-0,0006	-0,0137	100 L	0,0151	0,0023	0,0128
100 R	-0,0041	-0,0076	0,0035	100 R	0,0070	-0,0100	0,0170

vieillessement, que la représentation graphique de leurs évolutions en temps. De la liste résultante on a fait une présentation extrêmement sélective (Tableau 1) qui se trouve en corrélation directe avec la synthèse des nouveaux aspects révélés par l'étude accomplie. En dehors des coefficients $K T_y$ de vieillissement pour décades (les jours 1...10 et, respectivement, 10...100), on y trouve aussi une grandeur notée $K T_y$ ou A est définie comme il suit :

$$(1) \quad A \equiv \Delta K T_y = (K T_y)_{1-10} - (K T_y)_{10-100}.$$

Cette grandeur est une mesure de l'écart de la valeur moyenne $\bar{y}(10)$ évaluée dix jours après un traitement, par rapport à la droite déterminée par les points $\bar{y}(1)$ et $\bar{y}(100)$ (Fig. 1). On va la nommer l'écart de la linéarité. L'introduction de cette grandeur a été imposé par les raisons suivants :

1° On admet qu'une certaine grandeur, y , varie de façon logarithmique au cas où les coefficients de vieillissement pour les deux premières décades ont la même valeur : $(K T_y)_{1-10} = (K T_y)_{10-100}$; cela n'arrive, en réalité, qu'à très peu d'exception. Il se pose alors le problème quelle est la valeur maximale de la différence entre ces deux coefficients de vieillissement pour qu'on puisse considérer la variation à peu près logarithmique. Dans la littérature de spécialité il n'y a pas d'informations sur l'ordre de grandeur de cette différence. D'habitude, les catalogues donnent des courbes présentant l'évo-

lution des grandeurs dans le temps, représentées en coordonnées demilogarithmiques. L'appréciation de la variation logarithmique dans le temps dépend évidemment de l'échelle de la représentation et donc des possibilités d'estimation visuelle. Dans ces conditions, le critère nouveau introduit, l'écart de la linéarité A , représente un instrument beaucoup plus efficient dans le but proposé.

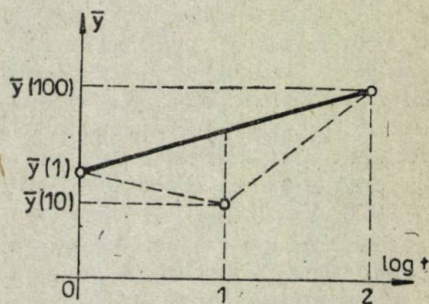


Fig. 1

2°. Sur la base de nos propres résultats expérimentaux, nous avons considéré qu'un écart de 1...2% de $\bar{y}(10)$ par rapport à la droite déterminée par $\bar{y}(1)$ et $\bar{y}(100)$ ne touche pas à la linéarité de l'évolution logarithmique dans le temps de la grandeur y (en coordonnées $y - \log t$). Donc on a considéré qu'une grandeur présente une variation dans le temps à peu près logarithmique d'après le critère de la valeur réduite (1...2%) de l'écart A de la linéarité.

En général les traitements thermiques servent à accélérer le vieillissement des matériaux et donc à l'établissement rapide de leurs paramètres. Poursuivant le processus de vieillissement accéléré, on peut dire que le matériau a un comportement prévisible dans le temps si les valeurs des coefficients de temps pour décades diminuent de façon monotone. Pourtant, quelquefois, on a constaté expérimentalement des évolutions anormales, exprimées par la relation suivante entre les coefficients de vieillissement pour décades

$$(2) \quad |KT_{y10 \dots 1000}| - |KT_{y1 \dots 10}| > 0.$$

Dans ces cas les évolutions ultérieures sont imprévisibles.

Les résultats obtenus peuvent être résumés ainsi :

a) en acceptant comme critère de la variations logarithmique des écarts de la linéarité inférieurs à 1%, seulement 57% des grandeurs diminuent spontanément de manière logarithmique (par vieillissement naturel) ;

b) parmi les traitements thermiques testés le traitement de revenu accompli à 60°C (60 L) a été trouvé optimal en ce qui concerne l'accroissement de l'activité piézoélectrique ;

c) la comparaison des traitements 80 A et 80 B a conduit à la conclusion suivante : après avoir subi un traitement de type 80B, seulement 15% des grandeurs répètent le schéma du vieillissement naturel. Par contre, après avoir subi un traitement de type 80A, 41% des grandeurs varient de façon similaire au vieillissement naturel. Donc il n'est pas vrai que „n'importe quel choc thermique conduit à la reprise du cycle de vieillissement naturel“.

3. Conclusions. 1. Parmi les traitements thermiques testés, ceux de trempe ont été trouvé peu favorables à la majorité des grandeurs d'intérêt (à l'exception de la constante de fréquence dans le mode radial de vibration N_p^x).

2. On peut prédire un comportement anormal après un certain traitement thermique, pour une grandeur quelconque, si est satisfaite la condition (2) entre les coefficients de vieillissement pour décades,

3. Consécutivement à un choc thermique produit plusieurs jours après le traitement de polarisation électrique, le schéma du vieillissement naturel est repris seulement au cas d'un nombre très réduit de grandeurs (15% pour notre matériau). Pour les autres grandeurs, le schéma du vieillissement est détérioré par rapport au cycle spontané, les valeurs des coefficients de vieillissement pour la deuxième décade étant en général plus grandes. Ce fait empêche la prédiction de l'évolution ultérieure.

4. Vu, d'une part, l'intérêt pratique pour l'établissement accéléré des paramètres, que, d'autre part, le fait que très peu des traitements expérimentés conduissent au but désiré, une étude systématique en vue de préciser le traitement approprié pour chaque nouveau matériau s'impose toujours, d'autant plus que la littérature de spécialité n'en offre pas d'informations.

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BIBLIOGRAPHIE

1. Jemna I. et al., Bul. Inst. Polit. Iași, **XXVIII (XXXII)**, 3-4, s. I, 97-101 (1982).
2. Antonescu I. et al., Les travaux de la Session jubilaire de communications scientifiques "175 Années d'Enseignement Technique en Roumain", Inst. Polytechnique de Jassy, 10-12 nov. 1988, Tome **III**, Section **VI**, pp. 28-31.
3. Jaffe B., Cook W. R. Jr., Jaffe H., *Piezoelectric Ceramics*. 1971.

ASPECTE NOI PRIVIND STABILITATEA ÎN TIMP A PIEZOCERAMICELOR CU ACTIVITATE PIEZOELECTRICĂ INTENSĂ

(Rezumat)

Se prezintă o metodologie experimentală completă pentru stabilizarea accelerată a parametrilor unui material piezoceramic cu activitate piezoelectrică intensă. Se formulează concluzii de utilitate practică, legate nemijlocit de material, precum și un criteriu obiectiv de predicție a evoluției pe termen lung a parametrilor oricărui material.

Profesor Dr. Docent **ALEXANDRU CLIMESCU**

(1910—1990)



It was on June 7, 1990, when the Romanian honoured man of science, D. Sc. Prof. Alexandru Climescu was passing away after a hard pain. His name was for decades closely connected with teaching mathematics in Jassy at the level of both technical and university higher education. Through his prodigious scientific activity carried on, he also distinguished himself as an outstanding representative of the mathematics research school in Jassy university centre.

Prof. Al. Climescu was born on May 11/24, 1910, in Bacău. He attended the Application High School in Jassy between 1921—1928 and then, between 1928—1932, started studying mathematics at the University „Al. I. Cuza“ of Jassy — Faculty of Science, taking his degree in Mathematics in 1933. He also took a degree in law after graduating the Faculty of Law, at the same university. In 1933—1934, he improved his teaching practice by attending the Pedagogic Seminar of the Superior Normal School in Jassy.

On November 1, 1934 he was appointed a professor's assistant at the Department of Function Theory — Section of Mathematics of the Faculty of Sciences where he worked until 1938, when he was promoted as a lecturer at another department — Higher Algebra — of the same faculty.

On June 15, 1940 he submitted his thesis for a Ph. D. degree entitled *Analytical Functions which Preserve the Halfplanes Determined by the Real Axis* at the University „Al. I. Cuza“ in Jassy before an examining board chaired by Academician Prof. Alexandru Myller himself — the founder of the Mathematical Seminary in Jassy.

Most of the between 1939 — 1945 he was called up in the army.

From 1945 to 1977 he worked as a professor at the Mathematics Department of the Polytechnic Institute of Jassy, teaching *Mathematical Analysis* at the Faculty of Electrical and Civil Engineering. From 1946 to 1967 he also taught *Special Chapters of Modern Algebra* and *Theory of Probabilities* as a professor at the Higher Algebra Department of the Faculty of Mathematics — University „Al. I. Cuza” in Jassy. By their high level and wide scientific knowledge covered, by their rigorous and original proofs, the courses taught by Prof. Climescu brought a higher standard in the development of the higher education in Jassy.

He was the head of the Mathematics Department of the Polytechnic Institute of Jassy from 1953 to 1958, Mathematics Department I (after the reorganization of the Institute) between 1958—1975. At the same time he led a section and a sector at the Mathematics Institute of the Romanian Academy — the Jassy branch.

In 1977, Prof. Al. Climescu retired from his teaching activity but remained a consulting professor at the department he was heading for years. He was a member of the Polytechnic Institute Senate and also a member of the Professors' Councils of the Electrical and Civil Engineering faculties, respectively.

In 1966 he was awarded as a Doctor Docent in Sciences (a scientific degree/title in Romania, next to the Ph. D. degree).

Professor Climescu was an outstanding contributor to the Bulletin of the Polytechnic Institute of Jassy, ever since it came into being in 1946. He also contributed to the Mathematical Reviews journal edited by A.M.S.

The high level and fruitful scientific activity prof. Climescu carried on came to be known through the publication of 71 papers. He took part in various scientific conferences, both in the country and abroad (U.S.S.R., Poland, Hungary, France).

He brought significant contributions to the following fields of mathematical research: *Theory of Functions of One Complex Variable*; *Modern Algebra*; *Arithmetics*; *Functional Equations*; *Classical and Modern Mathematical Analysis*; *Mathematical Programming*.

Starting from the previous researches of some famous mathematicians, he raises in his doctoral thesis the problem of finding out all the analytic functions that keep the halfplanes by introducing an operation denoted H (in the memory of Hurwitz and Hadamard), by means of which he characterizes the meromorphic functions.

In the field of Algebra he introduced the notion of index of nonassociativity and also the notions of group matrix weak ring, index for a Boolean algebra, complete sum for a semigroup. He showed that the theory of determinants becomes quicker and simpler if an axiomatic definition is adopted. Defining the weak ring as an algebraic structure similar to the rings, but with the property of distributivity replaced by certain relations, he proved a series of results on some classes of weak rings. He gave a unitary definition to the algebras with finitary operations and studied the linear spaces with positive scalars. He finds periodical solutions, belonging to any class C_p , for the integral equations

$$\int_0^{\infty} f(x^2 + y^2) dx = y, \quad -1 \leq y \leq 1,$$

and he characterizes the trigonometric functions by means of certain functional equations. He builds rather general solutions to the functional equations of the onesided distributivity and defines the class of countably linear multivalued functions. He proves the possibility of building up a calculus of everywhere defined mappings, analogous to the propositional calculus, and introduces a calculus of multivalued mappings of finite character.

In other works he defines a class of linear operators attached to numerical sequences indexed from $-\infty$ to $+\infty$, conventionally called operators of derivation, arriving to the construction of multivalued mappings defined on the set of rational numbers.

Among his results obtained in classical Mathematical Analysis, we are mentioning the one which proves the existence of the antiderivatives of a continuous function, starting from the Weierstrass-Bernstein approximation theorem, without using the concept of definite integral.

By using an elegant algebraic construction due to Al. Climescu, Romanian and foreign mathematicians have studied the fuzzy measures — mathematical objects of recent interest in the theory of Formal Languages, in Coding Theory and in the Theory of Games.

In the field of Mathematical Programming he proposed a linear programming method and introduced the notion of reverse programming, solving the problem thus stated in rather general conditions.

Prof. Al. Climescu's mathematical research work have met a wide interest among important mathematicians both from Romania and abroad. Suffice it to mention Tudora Luchian, Philip Holgate, E. P. Klement, J. L. Robinson.

The members of the Mathematics Department of the Polytechnic Institute of Jassy will never forget Professor Al. Climescu and will always recall his personality, his excellent qualities as a man, as a teacher and a scientist. We have learnt from our distinguished professor that life can be lived in a simple and lofty way with a little bit of wisdom and an open heart, that we can be happy if we work hard and passionately and that — although it sometimes requires great sacrifices — Mathematics can bring great satisfactions to those who are really endowed with the gift of serving it.

Department of Mathematics

RECENZII

BOOK REVIEWS

РЕЦЕНЗИИ

MICHAEL E. STARZAK, *Mathematical Methods in Chemistry and Physics*. Plenum Press, New York—London, 1989, 651 pp., ISBN 0-306-43066-5.

The book under review is a valuable course in matrix methods which can be used by the students and since it provides numerous examples to apply the mathematics in various areas of chemistry and physics, it can also be used as a supplementary text for courses in these areas. Although the author introduced only soluble $N \times N$ matrices which he considered that these are the most effective teaching tools, a student who develops a clear understanding of the basic physical systems presented in this book can easily extend his knowledge to more complicated systems which require numerical or computer techniques.

The book is divided into three sections and the author arranged the topics so chapters can be eliminated without disturbing the flow of information.

The first four chapters introduce the mathematics of vectors and matrices. Simple examples illustrate the basic concepts. Each new application then reinforces the applications which preceded it. Ch. 1, *Vectors*, introduces finite-dimensional vectors, components, the resolution of vectors and operations with vectors (bra-ket notation is introduced and used almost exclusively in subsequent chapters). For the triple scalar product is given a good example: a tetrahedral methane atom and the direct product appears frequently in quantum mechanics. Ch. 1 introduces concepts such as orthogonality, biorthogonality, projection operators and linear independence, vector calculus and applications. Ch. 2, *Function Spaces*, introduces function space vectors. To illustrate the strong parallels between such spaces and N -dimensional vector-spaces, the concepts of Ch. 1, e. g., orthogonality and linear independence, are developed for function space vectors. Projection techniques permit the description of any function in the space as a sum of orthogonal basis functions. In each case, the techniques of Ch. 1 serve as an heuristic guide. In many cases of interest here, the operators act on the function vector and some functions are studied under differential operation. Ch. 3 introduces *Matrices*, beginning with basic matrix operations and concluding with an introduction to eigenvalues and eigenvectors and their properties. The determinants are reviewed in a section of Ch. 3. The coordinate axes rotation technique is particularly valuable in quantum mechanics where rotation operations generate new orbitals from a single orbital. There are defined the principal axes or eigenvectors for the matrix. Ch. 4, *Similarity Transforms and Projections*, introduces practical techniques for the solution of matrix algebra and calculus problems. These include similarity transforms and projection operators. The chapter concludes with some finite difference technique for determining eigenvalues and eigenvectors for $N \times N$ matrices.

Chapters 5—8 apply the mathematics to the major areas of normal mode analysis, kinetics, statistical mechanics and quantum mechanics. Ch. 5, *Vibrations and Normal Modes*, gives a detailed descriptions of normal modes which requires solutions of the equations of motion for the atoms of the molecule. The techniques required are developed in this chapter. In Ch. 6, *Kinetics*, similar behavior is observed with systems of kinetic equations. Although distinct chemical species are normally monitored during kinetic experiments, the concentrations of these species are "blended" as linear combinations which correspond to the normal mode linear combinations in vibrating systems. The time evolution of distinct species is then easily separated from these combinations. Ch. 7, *Statistical Mechanics*, presents the introduction of transition probabilities describing a system which evolves to the proper equilibrium distribution, is irreversible in time and reduces the evolution of a large number of particles to a set of kinetic equations. The windtree model, the dog-flea model or the Ising model can also describe the evolution of the system to equilibrium. There are presented the transfer matrices for linear systems which

include residue-residue interactions which are not nearest neighbor, starting with an analysis of nearest and next-nearest-neighbor interactions. Ch. 8, *Quantum Mechanics*, starts with the remark that the matrices and the eigenvectors of the previous three chapters being finite-dimensional, some modifications are required in quantum mechanics. The wave functions of quantum mechanics are Hilbert space vectors with infinite, i. e., continuous components. This requires a modified point of view. It is given the Hyckel theory which restricts the basis set of eigenfunctions describing the wave properties of the electrons. An electron with two possible spin components generates Pauli spin matrices. In each chapter 5-8, the eigenvalues and eigenvectors for multicomponent coupled systems are related to familiar physical concepts.

The final three chapters introduce more advanced applications of matrices and vectors. Ch. 9, *Driven Systems and Fluctuations*, considers an alternate kinetic situation, for conservative systems, in which the state populations are generated by some input process. The ion flow through the channels of biological membranes can be described as a discrete state kinetic process. The nature of fluctuations about equilibrium is proved with a short-circuited capacitor and with a simple isomerization, observed when the system had reached equilibrium or during the evolution to equilibrium. Ch. 10, *Other Techniques; Perturbation Theory and Direct Products* acknowledges that the complicated matrices used to describe various physical phenomena can often be analyzed when their elements differ slightly from those of a second matrix whose eigenvalues are known. This is the purpose of perturbation theory. Ch. 11, *Introduction to Group Theory*, focuses on two main questions: (i) How are determined the eigenvectors for the symmetry elements in a symmetry space? (ii) How are determined the specific eigenvectors which generate valid molecular orbitals or normal modes? The group theory is introduced with an emphasis on the nature of matrices and vectors in this discipline.

This book is an extremely valuable one covering important aspects of the physical sciences because of the fortunate fact that the same mathematical ideas appear in a number of distinct scientific areas.

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